Matrix for Binomial Transform, backward difference: FIRST DRAFT

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INTRODUCTION

In this paper we present a new approach to evaluate sums concerning binomial transformations by using the matrix calculation. We can use these calculations rewriting the formulas given in paper of Boyadzhiev [1]. In particular we improve the calculation of this formula $\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{H_k}{k+x}$. In the appendix we give a new proof of A Reciprocal Summation Identity, [7]. We briefly outline the importance of the first coefficient a_0. In the last part of the appendix we show another proof for $p \leq n$, of the theorem Boyadzhiev [1], $\sum_{k=0}^{n}\binom{n}{k}k^p a_k = (n\bigtriangledown)^p b_n$, Some examples of this theorem are given.

Mathematical subject classificiation 2010: 05A10, 11B65,33C45.

Keywords Binomial transform, Backward difference operator, Harmonic numbers, Stirling numbers of the First and second kind, Fibonacci numbers, Complete homogeneous symmetric polynomials.

1 Backward difference operator

In this section we explain the multiplication of the one transformation binomial. We define the backward difference operator: $\triangledown b_k = b_k - b_{k-1}$, $k \in [1, 2, \dots, n]$ and we set the next matrices

$$P = \left(P_{ij}\right) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -2 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -n & n \end{pmatrix};$$

$$H = \left(\binom{n}{k}\right) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & n & \frac{n(n-1)}{2} & \cdots & n & 1 \end{pmatrix}; \quad D_0 = \left(d_{ij}\right) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & & n \end{pmatrix}$$

Lemma 1.1. *Let P, H, D_0 with as above, matrices of order $n+1$, then we have.*

$$PH = HD_0. \tag{1}$$

Proof. We have $s\binom{r}{s} = r\binom{r-1}{s-1} = r\binom{r}{s} - r\binom{r-1}{s}$ and because the $rs-$component in the RHS in (1) is $\sum_{j=0}^{n} \binom{r}{j} d_{js} = s\binom{r}{s}$ and the $rs-$component of the LHS in (1) is $\sum_{j=0}^{n} p_{rj}\binom{j}{s} = r\binom{r}{s} - r\binom{r-1}{s}$, therefore we have (1). □

Remark 1.1. *Because the matrix H is invertible we have the relation.*

$$\sum_{k=0}^{n} k\binom{l}{k}(-1)^{k-j}\binom{k}{j} = \begin{cases} l & if \quad j = l; \\ -l & if \quad j-1 = l; \\ 0 & \text{the other case.} \end{cases}$$

This formula is similar to the formula 3.119 in Gould ([3]).

Remark 1.2. *The equation (1) we tells us that the matrices P, D_0 are similar therefore*

$$P^r H = HD_0^r, \quad \forall \quad r \in \mathbb{N}.$$

Theorem 1.1. *Let $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$. For every positive integer p and $n \geq p$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} k^p a_k = (n\triangledown)^p b_n. \tag{2}$$

Here we need $n \geq p$ in the above formula because the RHS contains a term with b_{n-p} which is not defined for $n < p$. It's clear that we only need to prove (2) for $p = 1$ and the rest follows by iteration.

Proof. We give the prove for $p = 1$. i.e. $\sum_{k=0}^{n} \binom{n}{k} k a_k = (n\bigtriangledown) b_n = n(b_n - b_{n-1})$. We write the LHS of (2) in the form of vector column.

$$\begin{pmatrix} \sum_{k=0}^{0} \binom{0}{k} k a_k \\ \sum_{k=0}^{1} \binom{1}{k} k a_k \\ \sum_{k=0}^{1} \binom{2}{k} k a_k \\ \vdots \\ \sum_{k=0}^{n} \binom{n}{k} k a_k \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & n & \frac{n(n-1)}{2} & \cdots & n & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & & n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -2 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -n & n \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & n & \frac{n(n-1)}{2} & \cdots & n & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -2 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -n & n \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} 0(b_0 - 0) \\ 1(b_1 - b_0) \\ 2(b_2 - b_1) \\ \vdots \\ n(b_n - b_{n-1}) \end{pmatrix} = \begin{pmatrix} 0 \bigtriangledown b_0 \\ 1 \bigtriangledown b_1 \\ 2 \bigtriangledown b_2 \\ \vdots \\ n \bigtriangledown b_n \end{pmatrix}$$

This prove the equation (2) for $p = 1$. □

2 DIVISION I

In this section we prove the division of a binomial transform. We set the next matrices

$$D_1 = (d_{ij}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & & \frac{1}{n} \end{pmatrix}; \quad H_n = (h_{ij}) = \left(\binom{i}{j}\right) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 0 & \cdots & 0 & 0 \\ 3 & 3 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n & \frac{n(n-1)}{2} & \binom{n}{3} & \cdots & n & 1 \end{pmatrix};$$

$$H = (b_{ij}) = \left(\binom{i}{j}\right) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & n & \frac{n(n-1)}{2} & \cdots & n & 1 \end{pmatrix}; U = (u_{ij}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & & 1 \end{pmatrix}.$$

Proposition 2.1. *Let U, H, D_1 matrices of order $n+1$, with as above then we have*

$$D_1 U H = H D_1. \tag{3}$$

Proof. The $rc-$component with $0 \le r, c \le n$, of LHS of (3) is

$$\sum_{k=0}^{n}\sum_{l=0}^{n} d_{rk} u_{kl} b_{lc} = \sum_{l=0}^{r} \frac{1}{r+1}\binom{l}{c} = \frac{1}{r+1}\sum_{l=0}^{r}\binom{l}{c} = \frac{1}{r+1}\binom{r+1}{c+1} = \frac{1}{c+1}\binom{r}{c},$$

that element is the $rc-$component with $0 \le r, c \le n$ of RHS of (3). □

we have that the matrices D_1U and D_1 are similar therefore

$$(D_1U)^p H = H D_1^p.$$

Corollary 2.1. *We have*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{k+1} = \frac{1}{n+1}\sum_{m=0}^{n} b_m. \tag{4}$$

Proof. let $A(n) = (a_0, a_1, a_2, \cdots, a_n)^T$, $B(n) = (b_0, b_1, b_2, \cdots, b_n)^T$ and $HA(n) = B(n)$

The LHS of (4) is the last column vector component of

$$HD_1A(n) = D_1UHA(n) = D_1UB(n)$$

that is none other the RHS of (4). □

Proposition 2.2. *Let U, H_n, D_1 matrices of order n, with as above then we have this relation*

$$UD_1H_n = H_nD_1. \tag{5}$$

Proof. Let $1 \le r, c \le n$. the $rc-$component of LHS (5) is

$$\sum_{k=1}^{n}\sum_{l=1}^{n} u_{rk}d_{k,l}h_{lc} = \sum_{k=1}^{r}\frac{1}{k}\binom{k}{c} = \sum_{k=1}^{r}\frac{1}{c}\binom{k-1}{c-1} = \frac{1}{c}\sum_{k=1}^{r}\binom{k-1}{c-1} = \frac{1}{c}\binom{r}{c},$$

that element is the $rc-$component of RHS of (5) then the proposition is demonstrated. □

The formula (5) proves that the matrices UD_1 and D_1 are similar so we have this relation

$$(UD_1)^pH_n = H_nD_1^p. \tag{6}$$

Corollary 2.2. *We set $a_0 = b_0 = 0$. Then*

$$\sum_{k=1}^{n}\binom{n}{k}\frac{a_k}{k} = \sum_{m=1}^{n}\frac{b_m}{m}. \tag{7}$$

Proof. let $A(n) = (a_1, a_2, a_3, \cdots, a_n)^T$, $B(n) = (b_1, b_2, b_3, \cdots, b_n)^T$ and $H_nA(n) = B(n)$

The LHS of (7) can be written as the last component of the column vector of

$$H_nD_1A(n) = UD_1H_nA(n) = UD_1B(n)$$

that is none other the RHS of (7). □

Example 2.1. *We set $A(n) = (a_n) = (1, -1, \cdots, (-1)^{n-1}) \in \mathbb{Z}^n$; $B(n) = (b_n) = (1, \cdots, 1) \in \mathbb{Z}^n$ and $a_0 = 0, b_0 = 0$. Then we have*

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k} = \sum_{m=1}^{n}\frac{1}{m} =: H_n. \tag{8}$$

and by inversion we have

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}H_k = \frac{1}{n}. \tag{9}$$

Proof. The LHS of (8) can be written as the last component of column vector of $H_nD_1A(n) = UD_1H_nA(n) = UD_1B(n)$ that is none other than the RHS of (8). □

Example 2.2. *We set $H(n) = (H_1, H_2, \cdots, H_n)$. We have this relation:*

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k^2} = \sum_{m=1}^{n}\frac{H_m}{m}. \tag{10}$$

Proof. In this case we use the relation $(UD_1)(UD_1)H_n = (UD_1)^2 H_n = H_n D_1^2$.

The LHS of (10) can be written as the last term of column vector of

$$H_n D_1^2 A(n) = UD_1 UD_1 H_n A(n) = UD_1 UD_1 B(n) = UD_1 H(n)$$

that element is the RHS of (10). □

3 Power

In this part we provide a formula to find a power of lower trianguler matrix i.g.

$$A = \begin{pmatrix} R(1,1) \\ R(2,1) \\ R(3,1) \\ \vdots \\ R(n,1) \end{pmatrix} := \begin{pmatrix} x_1 & 0 & 0 & \cdots & 0 & 0 \\ x_1 & x_2 & 0 & \cdots & 0 & 0 \\ x_1 & x_2 & x_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & & 1 \end{pmatrix} \begin{pmatrix} x_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & x_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & & x_n \end{pmatrix} = UD_x.$$

and we set $R(i,p)$ the i row vector of A^p.

and we use the complete homogeneous symmetric polynomial of degree k in n variables $x_1, x_2 ..., x_n$, written h_k for $k = 0, 1, 2, ...$, is the sum of all monomials of total degree k in the variables. Formally,

$$h_k(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & if \quad k = 0; \\ \displaystyle\sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}, & if \quad k \ge 1. \end{cases}$$

and we have the recurrence relation for the Complete homogeneous symmetric polynomial

$$h_p(x_1,x_2,\ldots,x_n) = \sum_{1\le \underbrace{i_1 \le i_2 \le \cdots \le i_k}_{p-elements}\le n} x_{i_1}x_{i_2}\cdots x_{i_k} =$$

$$\sum_{l=1}^{n} x_l \sum_{1\le \underbrace{i_1 \le i_2 \le \cdots \le i_{k-1}}_{(p-1)-elements}\le l} x_{i_1}x_{i_2}\cdots x_{i_{k-1}} = \sum_{l=1}^{n} x_l h_{p-1}(x_1,x_2,\ldots,x_l). \tag{11}$$

Lemma 3.1.

$$A^p(1,1,\cdots,1)^T = (h_p(x_1), h_p(x_1,x_2),\cdots,h_p(x_1,x_2,\cdots,x_n))^T \tag{12}$$

Proof. We use induction on p. It is clear that the formula (12) is true for $p=1$. Then we shall show that the formula holds for $p+1$ and we assume that it holds for p, by using (11) we have

$$A^{p+1}(1,1,\cdots,1)^T = AA^p(1,1,\cdots,1)^T = A(h_p(x_1),h_p(x_1,x_2),\cdots,h_p(x_1,x_2,\cdots,x_n))^T =$$

$$UD_x(h_p(x_1),h_p(x_1,x_2),\cdots,h_p(x_1,x_2,\cdots,x_n))^T = (h_{p+1}(x_1),h_{p+1}(x_1,x_2),\cdots,h_{p+1}(x_1,x_2,\cdots,x_n))^T. \tag{13}$$

□

Theorem 3.1. *Let $p \ge 1$. We set $A^p(i,j)$ the ij-component of A^p. Then the lower triangular matrix is*

$$A^p(i,j) = \begin{cases} 0 & if \quad j > i; \\ h_{p-1}(x_j,x_{j+1},\ldots,x_i)x_j & if \quad i \ge j. \end{cases}$$

Proof. The proof is by induction. For $p=1$ is the definition. We prove for $p=2$, because we see the induction step, and we do the calculation only in the first column vector of A^2.

$$A(x_1,x_1,\ldots,x_1)^T = x_1A(1,1,\ldots,1)^T = (x_1h_1(x_1),x_1h_1(x_1,x_2),\cdots,x_1h_1(x_1,x_2,\cdots,x_n))^T \tag{14}$$

by induction and by relation (13) we have.

$$A^{p+1} = A(x_1h_{p-1}(x_1),x_1h_{p-1}(x_1,x_2),\cdots,x_1h_{p-1}(x_1,x_2,\cdots,x_n))^T =$$

$$x_1A(h_{p-1}(x_1),h_{p-1}(x_1,x_2),\cdots,h_{p-1}(x_1,x_2,\cdots,x_n))^T = x_1(h_p(x_1),h_p(x_1,x_2),\cdots,h_p(x_1,x_2,\cdots,x_n))^T.$$

□

Example 3.1. *We set* $A(n) = (a_n) = (1, -1, \cdots, (-1)^{n-1}) \in \mathbb{Z}^n$*;* $B(n) = (b_n) = (1, \cdots, 1) \in \mathbb{Z}^n$ *and* $a_0 = 0, b_0 = 0$*. Then we have*

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k^2} = \sum_{1 \le k_1 \le k_2}^{n} \frac{1}{k_1 k_2}. \tag{15}$$

Proof. In this case we define the sequence of numbers x_i by $x_i = \frac{1}{i}$. By using the notation of Section 2 and the formula (14), then we have this relation:

The LHS of (15) can be written as the last term of column vector of $H_n D_1^2 A(n) = (UD_1)^2 H_n A(n) = (UD_1)^2 B(n) = (UD_1)^2 (1, 1, \cdots, 1)^t$ that is none other the RHS of (15).

□

Example 3.2. *We have this relation:*

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k^m} = \sum_{1 \le k_1 \le k_2 \le k_3 \le \cdots \le k_m \le n}^{n} \frac{1}{k_1 k_2 k_3 \cdots k_m}. \tag{16}$$

Proof. The LHS of (16) can be written as the last term of column vector of $H_n D_1^m A(n) = (UD_1)^m H_n A(n) = (UD_1)^m B(n) = (UD_1)^m (1, 1, \cdots, 1)^t$ that is the RHS of (16). □

Those examples were obtained, in different way, by K. Dilcher [2] and by K- N. Boyadzhiev [1].

4 DIVISION II

We call the following matrices of order n.

$$D_x = \begin{pmatrix} 1+x & 0 & 0 & \cdots & 0 & 0 \\ -2 & 2+x & 0 & \cdots & 0 & 0 \\ 0 & -3 & 3+x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -n & n+x \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -2 & 2 & 0 & \cdots & 0 & 0 \\ 0 & -3 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -n & n \end{pmatrix} + \begin{pmatrix} x & 0 & 0 & \cdots & 0 & 0 \\ 0 & x & 0 & \cdots & 0 & 0 \\ 0 & 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & x \end{pmatrix}$$

$$\overline{D_x} = (1+x, 2+x, \cdots, n+x) = \begin{pmatrix} 1+x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2+x & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3+x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n+x \end{pmatrix};$$

$$H_n = \left(\binom{n}{k}\right) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 0 & \cdots & 0 & 0 \\ 3 & 3 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n & \frac{n(n-1)}{2} & \binom{n}{3} & \cdots & n & 1 \end{pmatrix};$$

Proposition 4.1. *let $D_x, H_n, \overline{D}_x$ with as above then we have*

$$D_x H_n = H_n \overline{D_x}. \tag{17}$$

Proof. In this proof we are going to transform D_x in a sum of two matrices diagonalizable for the same matrix H_n i.e. the first is a square submatrix of matrix P of Lemma (1.1) the second is a scalar matrix.

□

Theorem 4.1. *We assume $x_i \neq 0$ and we define the next lower triangular matrix of orden n,*

$$D_x = \begin{pmatrix} x_1 & 0 & 0 & \cdots & 0 & 0 \\ -2 & x_2 & 0 & \cdots & 0 & 0 \\ 0 & -3 & x_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -n & x_n \end{pmatrix}$$ *; then the inverse matrix D_x^{-1} is*

$$D_x^{-1} = \begin{pmatrix} \frac{1}{x_1} & 0 & 0 & \cdots & 0 & 0 \\ \frac{2}{x_1x_2} & \frac{1}{x_2} & 0 & \cdots & 0 & 0 \\ \frac{6}{x_1x_2x_3} & \frac{3}{x_2x_3} & \frac{1}{x_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(n-1)!}{x_1x_2\cdots x_{n-1}} & \frac{(n-1)!x_1}{2x_1x_2\cdots x_{n-1}} & \frac{(n-1)!x_1x_2}{3!x_1x_2\cdots x_{n-1}} & \cdots & \frac{(n-1)!x_1\cdots x_{n-3}x_{n-2}}{(n-1)!x_1\cdots x_{n-2}x_{n-1}} & 0 \\ \frac{n!}{x_1x_2\cdots x_n} & \frac{n!x_1}{2x_1x_2\cdots x_n} & \frac{n!x_1x_2}{3!x_1x_2\cdots x_n} & \cdots & \frac{n!x_1\cdots x_{n-3}x_{n-2}}{(n-1)!x_1\cdots x_{n-1}x_n} & \frac{n!x_1\cdots x_{n-2}x_{n-1}}{n!x_1\cdots x_{n-1}x_n} \end{pmatrix} \tag{18}$$

Proof. Is very simple to do that $D_x D_x^{-1} = I_n$ □

Proposition 4.2. *We assume $x \in \mathbb{N}$, $x_0 = 1, x_1 = 1 + x, x_2 = 2 + x, \cdots, x_n = n + x$, $\left[{a \atop b}\right]$ The Stirling numbers of the first kind unsigned. Then the n-row of matrix D_x^{-1} is:*

$$\left(\frac{n!}{x_1x_2\cdots x_n}, \quad \frac{n!x_1}{2x_1x_2\cdots x_n}, \quad \frac{n!x_1x_2}{3!x_1x_2\cdots x_n}, \quad \cdots \quad, \frac{n!x_1\cdots x_{n-2}x_{n-1}}{n!x_1\cdots x_{n-1}x_n}\right) =$$

$$\left(\frac{\sum_{l=0}^{x-1}\begin{bmatrix} x \\ l+1 \end{bmatrix}1^l}{(n+1)(n+2)\cdots(n+x)}, \quad \frac{\sum_{l=0}^{x-1}\begin{bmatrix} x \\ l+1 \end{bmatrix}2^l}{(n+1)(n+2)\cdots(n+x)}, \quad \frac{\sum_{l=0}^{x-1}\begin{bmatrix} x \\ l+1 \end{bmatrix}3^l}{(n+1)(n+2)\cdots(n+x)}, \quad ,\cdots, \frac{\sum_{l=0}^{x-1}\begin{bmatrix} x \\ l+1 \end{bmatrix}n^l}{(n+1)(n+2)\cdots(n+x)}\right) \tag{19}$$

Proof. The proof is divided in two steps: the first one by using the condition $n \geq x \geq 1$ and the second one $x \geq n$.

1° Step. Let $n \geq x \geq 1$. We take the k component of LHS of (19) and by using the rising factorial we have.

$$\frac{n!x_1x_2\cdots x_{k-1}}{k!x_1x_2\cdots x_n} = \frac{n!\cdot 1\cdot 2\cdots x\cdot(x+1)\cdot(x+2)\cdots(x+k-1)}{k!\cdot 1\cdot 2\cdots x\cdot(x+1)(x+2)\cdots(x+n)} = \frac{n!\cdot 1\cdot 2\cdots x\cdot(x+1)\cdot(x+2)\cdots(x+k-1)}{k!\cdot 1\cdot 2\cdots x\cdot(x+1)(x+2)\cdots(x+n-x)\cdot(x+n-x+1)\cdots(x+n)}$$

$$= \frac{(x+k-1)!}{k!}\frac{1}{(n+1)(n+2)\cdots(x+n)} = \sum_{l=0}^{x-1}\begin{bmatrix} x \\ l+1 \end{bmatrix}k^l\frac{1}{(n+1)(n+2)\cdots(x+n)}$$

That is the k component of RHS of (19).

2° Step. Let $x \geq n \geq 1$. We take the k component of LHS of (19) and by using the rising factorial we have.

$$\frac{n!x_1x_2\cdots x_{k-1}}{k!x_1x_2\cdots x_n} = \frac{n!(n+1)\cdot(n+2)\cdots(n+x-n)(1+x)(2+x)\cdots(k-1+x)}{k!(n+1)\cdot(n+2)\cdots(n+x-n)(1+x)(x+2)\cdots(n+x)} = \frac{(k-1+x)!}{k!}\frac{1}{(n+1)\cdot(n+2)\cdots(n+x-n)(1+x)(x+2)\cdots(n+x)}$$

$$= \sum_{l=0}^{x-1}\begin{bmatrix} x \\ l+1 \end{bmatrix}k^l\frac{1}{(n+1)(n+2)\cdots(x+n)}$$

That is the k component of RHS of (19).

□

Example 4.1. *let* $A(n) = (a_1, a_2, a_3, \cdots, a_n)^T$, $B(n) = (b_1, b_2, b_3, \cdots, b_n)^T$ *and* $H_nA(n) = B(n)$ *then*

$$\sum_{k=1}^{n}\binom{n}{k}\frac{a_k}{k+1} = \frac{1}{n+1}\sum_{m=1}^{n}b_m. \tag{20}$$

Proof. In the equation (17) we set $x = 1$ therefore $x_1 = 2, x_2 = 3, \cdots, x_n = n+1$.

We call $\overline{D_1}(2, 3, \cdots, n+1)$ the diagonal matrix then using the formula (18) we have

$$\overline{D_1}^{-1} = \overline{D_1}^{-1}(2,3,\cdots,n+1)^T = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \cdots & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \cdots & & \frac{1}{n+1} \end{pmatrix}$$

The LHS of (20) can be written as the last component of column vector

$$(\binom{n}{k}\frac{a_k}{k+1}) = H_n\overline{D_x}^{-1}A(n)^T = H_n\overline{D_x}^{-1}H_n^{-1}B(n)^T = \overline{D_1}^{-1}(2,3,4,\cdots,n+1)B(n)^T \qquad (21)$$

that is none other than the RHS of (20). □

5 APPLICATION

Theorem 5.1. *Let* $1 \le x \in N$. *We set* $\left[{a \atop b}\right]$ *The Stirling numbers of the first kind unsigned.* $Z[n,r] = \sum_{l=1}^{n} l^r$ *and we set:* $\sum_{l=1}^{0}\left[{x \atop l+1}\right]Z[n,l-1] = 0$.
$A(n) = (a_n) = (H_1, -H_2, \cdots, (-1)^{n-1}H_n) \in \mathbb{Q}^n$*;* $B(n) = (b_n) = (1, \frac{1}{2}\cdots, \frac{1}{n})$ *and* $a_0 = 0, b_0 = 0$. *Then we have this formula.*

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{H_k}{k+x} = \frac{\sum_{l=1}^{x-1}\left[{x \atop l+1}\right]Z[n,l-1] + (x-1)!H_n}{(n+1)(n+2)\cdots(n+x)}. \qquad (22)$$

Remark We can use Faulhaber's formula to obtain zeta function.

Proof. :We give the proof in two steps, the firstly using the condition $x = 1$, and the secondly using the condition $x \neq 1$.

By using the relation (19) the LHS of (22) is

$$(\binom{n}{k}\frac{(-1)^{k-1}H_k}{k+x}) = H_n\overline{D_x}^{-1}A(n)^T = H_n\overline{D_x}^{-1}H_n^{-1}B(n)^T = D_x^{-1}B(n)^T \qquad (23)$$

1° Step. Let $x = 1$, in this case the formula (22) is

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{H_k}{k+1} = \frac{H_n}{(n+1)}. \qquad (24)$$

This formula (24) is the formula (20) the example(4.1).

2° Step. Let $x \neq 1$, and $x_0 = 1, x_1 = 1 + x, x_2 = 2 + x, \cdots, x_n = n + x$. Let $k \in [1, 2, 3 \cdots, n]$. By the Proposition (5.1) the k component of the last row of matrix (18) is

$$\frac{1}{(n+1)(n+2)(n+3)\cdots(n+x)} \sum_{l=0}^{x-1} \begin{bmatrix} x \\ l+1 \end{bmatrix} k^l = \frac{1}{(n+1)(n+2)(n+3)\cdots(n+x)} \left(\sum_{l=1}^{x-1} \begin{bmatrix} x \\ l+1 \end{bmatrix} k^l + (x-1)! \right)$$

The RHS of equation(23) is

$$\sum_{k=1}^{n} \frac{\sum_{l=1}^{x-1} \begin{bmatrix} x \\ l+1 \end{bmatrix} k^l + (x-1)!}{(n+1)(n+2)(n+3)\cdots(n+x)} \frac{1}{k} = \frac{\sum_{l=1}^{x-1}\sum_{k=1}^{n} \begin{bmatrix} x \\ l+1 \end{bmatrix} k^{l-1} + (x-1)! H_n}{(n+1)(n+2)(n+3)\cdots(n+x)}$$ this the RHS of equation (22).

□

Example 5.1. *We have this relation for $x = 2$:*

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{H_k}{k+2} = \frac{n + H_n}{(n+1)(n+2)} \tag{25}$$

Example 5.2. *We have this relation for $x = 3$:*

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{H_k}{k+3} = \frac{n^2 + 7n + 4H_n}{2(n+1)(n+2)(n+3)}. \tag{26}$$

Example 5.3. *We have this relation for $x = 4$:*

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{H_k}{k+4} = \frac{2n^3 + 21n^2 + 85n + 36H_n}{6(n+1)(n+2)(n+3)(n+4)}. \tag{27}$$

Those examples were obtained by Juneseng Choi and H.M.Srivastava [5] and by Khristo N. Boyadzhiev [1] by the other means.

Example 5.4. *We have this relation for $x = 5$, this case in new. :*

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{H_k}{k+5} = \frac{3n^4 + 40n^3 + 279n^2 + 830n + 288H_n}{12(n+1)(n+2)(n+3)(n+4)(n+5)}. \tag{28}$$

6 Appendix

Problem 10490: In this section we give another proof of problem 10490 [7]. As you can see in the proof the first elementa_0 plays an important role

Theorem 6.1. *Let $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$. For any $n \geq 0$, we have*

$$\sum_{k=1}^{n} \frac{b_k}{k} = \sum_{k=1}^{n} \binom{n}{k} \frac{a_k}{k} + a_0 \sum_{l=1}^{n} \frac{1}{l}. \tag{29}$$

Boyadzhiev [1] has another proof.

Proof. $$\sum_{l=1}^{n}\frac{b_l}{l}=\sum_{l=1}^{n}\frac{\sum_{k=0}^{l}\binom{l}{k}a_k}{l}=\sum_{l=1}^{n}\sum_{k=0}^{l}\frac{\binom{l}{k}a_k}{l}=\sum_{l=1}^{n}\sum_{k=0}^{n}\frac{\binom{l}{k}a_k}{l}=\sum_{k=0}^{n}\sum_{l=1}^{n}\frac{\binom{l}{k}a_k}{l}=$$
$$=\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{\binom{l}{k}a_k}{l}+\sum_{l=1}^{n}\frac{\binom{l}{0}a_0}{l}=\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{\frac{l}{k}\binom{l-1}{k-1}a_k}{l}+\sum_{l=1}^{n}\frac{a_0}{l}=$$
$$=\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{\binom{l-1}{k-1}a_k}{k}+\sum_{l=1}^{n}\frac{a_0}{l}=\sum_{k=1}^{n}\sum_{l=0}^{n-1}\frac{\binom{l}{k-1}a_k}{k}+\sum_{l=1}^{n}\frac{a_0}{l}=$$
$$=\sum_{k=1}^{n}\frac{a_k}{k}\sum_{l=0}^{n-1}\binom{l}{k-1}+a_0\sum_{l=1}^{n}\frac{1}{l}=\sum_{k=1}^{n}\frac{a_k}{k}\binom{n}{k}+a_0\sum_{l=1}^{n}\frac{1}{l}=\sum_{k=1}^{n}\frac{a_k}{k}\binom{n}{k}+a_0H_n.$$ □

In this appendix we do another presentation, for $p\le n$, of the theorem Boyadzhiev[1] $\sum_{k=0}^{n}\binom{n}{k}k^pa_k=(n\bigtriangledown)^pb_n$. The proof uses the next lemma:

Lemma 6.1. *Let $p\le n$. Then :*

$$\binom{n}{k}k^p=\sum_{l=0}^{p}\sum_{j=0}^{p}(-1)^l\binom{n-l}{k}\binom{n}{l}\binom{n-l}{j-l}j!\left\{{p\atop j}\right\}. \tag{30}$$

Proof. The RHS of (30) is the same of
$$\sum_{l=0}^{p}\sum_{j=0}^{p}(-1)^l\binom{n}{k}\binom{n-k}{l}\binom{n-l}{j-l}j!\left\{{p\atop j}\right\}=\binom{n}{k}\sum_{l=0}^{p}\sum_{j=0}^{p}(-1)^l\binom{n-k}{l}\binom{n-l}{j-l}j!\left\{{p\atop j}\right\}.$$
We proof the next relation $\sum_{l=0}^{p}(-1)^l\binom{n-k}{l}\binom{n-l}{j-l}=\binom{k}{j}$, because $l\le j$ we have:
$$\sum_{l=0}^{p}(-1)^l\binom{n-k}{l}\binom{n-l}{j-l}=\sum_{l=0}^{j}(-1)^l\binom{n-k}{l}\binom{n-l}{j-l}=\sum_{l=0}^{j}\binom{-n+k+l-1}{l}\binom{n-l}{j-l}.$$
We set $x=k-n-1,\quad y=n-j$ by Gould[2] pag 22. We have.
$\sum_{l=0}^{j}\binom{-n+k+l-1}{l}\binom{n-l}{j-l}=\sum_{l=0}^{j}\binom{x+l}{l}\binom{y+j-l}{j-l}=\binom{x+y+j+1}{j}=\binom{k}{j}$. Therefore The RHS of (30) is $\binom{n}{k}k^p$. □

Here follow some examples of lemma(6.1).

$$\binom{n}{k}k^p=\begin{cases} n^2\binom{n}{k}-n(2n-1)\binom{n-1}{k}+n(n-1)\binom{n-2}{k}, & \text{if}\quad p=2;\\ n^3\binom{n}{k}-n(3n^2-3n+1)\binom{n-1}{k}+3n(n-1)^2\binom{n-2}{k}-n(n-1)(n-2)\binom{n-3}{k}, & \text{if}\quad p=3.\\ n^4\binom{n}{k}-n(4n^3-6n^2+4n-1)\binom{n-1}{k}+n(6n^3-18n^2+19n-7)\binom{n-2}{k}-n(4n^3-18n^2+26n-12)\binom{n-3}{k}+ & \\ n(n-1)(n-2)(n-3)\binom{n-4}{k}, & \text{if}\quad p=4;\\ n^p\binom{n}{k}+n(n^p-(n-1)^p)\binom{n-1}{k}+\cdots+(-1)^p\binom{n}{p}p!\binom{n-p}{k}, & \text{for any}\quad p\le n; \end{cases}$$

Theorem 6.2. *Let $b_n=\sum_{k=0}^{n}\binom{n}{k}a_k$. For every positive integer p and $n\ge p$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}k^pa_k=\sum_{k=0}^{n}\sum_{l=0}^{p}\sum_{j=0}^{p}(-1)^l\binom{n-l}{k}\binom{n}{l}\binom{n-l}{j-l}j!\left\{{p\atop j}\right\}a_k=(n\bigtriangledown)^pb_n. \tag{31}$$

Remark 6.1. *we set* $P = \begin{pmatrix} P_{ij} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -2 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -n & n \end{pmatrix}$, $1 \leq p \leq n$ *and* $R(n,p)$ *the* n *row of matrix* P^p.$\forall\, l \in [0, 1, \cdots, p]$ *and by using (1.2), (2) and (31) we have .*

$$R(n,p) = \left(\underbrace{0, \cdots, 0,}_{n-p} \underbrace{(-1)^p \binom{n}{p} p!,}_{l=0} \cdots \sum_{j=0}^{p} (-1)^{p-l} \binom{n}{p-l} \binom{n-p+l}{n-j} j! \left\{ {p \atop j} \right\}, \cdots, \underbrace{n^p}_{l=p} \right)$$

7 Examples

In this section we shall use the above lemma in order to obtain a new proof of examples of Boyadzhiev [1] and Spivey [6].

Example 7.1. *Spivey* [6]. *Let* $\{a_n = 1\}$ *and* $\{b_n = 2^n\}$ *sequences with property* $\sum_{k=0}^{n} \binom{n}{k} a_k = b^k$. *Let* $m \leq n$ *then*

$$\sum_{k=0}^{n} \binom{n}{k} k^m = \sum_{l=0}^{m} \binom{n}{l} \left\{ {m \atop l} \right\} l! 2^{n-l} \tag{32}$$

Proof. By (31) the LHS (32) is

$$\sum_{k=0}^{n} \sum_{l=0}^{m} \sum_{j=0}^{m} (-1)^l \binom{n-l}{k} \binom{n}{l} \binom{n-l}{j-l} j! \left\{ {m \atop j} \right\} = \sum_{l=0}^{m} \sum_{j=0}^{m} (-1)^l 2^{n-l} \binom{n}{l} \binom{n-l}{j-l} j! \left\{ {m \atop j} \right\} =$$

$$\sum_{l=0}^{m} \sum_{j=0}^{m} (-1)^l 2^{n-l} \binom{n}{n-l} \binom{n-l}{n-j} j! \left\{ {m \atop j} \right\} = \sum_{l=0}^{j} \sum_{j=0}^{m} (-1)^l 2^{n-l} \binom{n}{j} \binom{j}{j-l} j! \left\{ {m \atop j} \right\} =$$

$$\sum_{j=0}^{m} 2^{n-j} \sum_{l=0}^{j} (-1)^l 2^{j-l} \binom{j}{l} \binom{n}{j} j! \left\{ {m \atop j} \right\} = \sum_{j=0}^{m} \binom{n}{j} \left\{ {m \atop j} \right\} j! 2^{n-j}$$

□

Example 7.2. *Let* $a_k = x^k$, *where* x *is any real or complex number. Then* $b_n = -(1-x)^n$.

We have this relation in our case $p \leq n$.

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k k^p x^k = \sum_{j=0}^{p} \binom{n}{j} \left\{ {p \atop j} \right\} j! (-1)^j x^j (1-x)^{n-j} \tag{33}$$

Proof. The RHS of (33) is the same of this :

$$\sum_{j=0}^{p} \sum_{l=0}^{j} \binom{n}{j} \binom{j}{l} \left\{ {p \atop j} \right\} j! (-1)^l (1-x)^{n-l} = \sum_{j=0}^{p} \sum_{l=0}^{p} \binom{n}{l} \binom{n-l}{j-l} \left\{ {p \atop j} \right\} j! (-1)^l (1-x)^{n-l}.$$

By the lemma the LHS of (33) is :

$$\sum_{k=0}^{n}\sum_{l=0}^{p}\sum_{j=0}^{p}(-1)^k(-1)^l\binom{n-l}{k}\binom{n}{l}\binom{n-l}{j-l}j!\left\{{p \atop j}\right\}x^k=\sum_{l=0}^{p}\sum_{j=0}^{p}(-1)^l\binom{n}{l}\binom{n-l}{j-l}j!\left\{{p \atop j}\right\}(1-x)^{n-l}. \qquad \square$$

Remark 7.1. *Of course, for $x=-1$ the equation (33) is the same of equation (32).*

Example 7.3. *Let $a_k = H_k = 1+\frac{1}{2}+\cdots+\frac{1}{k}$, where $k>0, a_0=0, p\le n$. Then $b_0=0$ and for $n>0$ we have $b_n=\frac{-1}{n}$, then we have a formula:*

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}H_k k^p = (-1)^n n!\left\{{p \atop n}\right\}H_n - \sum_{k=1}^{p-1}\frac{(-1)^k k!\left\{{p \atop k}\right\}}{n-k} \tag{34}$$

Proof. Let $p<n$, then the LHS of (34) is
$\sum_{l=1}^{p}\sum_{j=0}^{p}(-1)^l\frac{-1}{n-l}\binom{n}{l}\binom{n-l}{j-l}j!\left\{{p \atop j}\right\}=\sum_{l=1}^{p}\sum_{j=0}^{p}(-1)^l\frac{-1}{n-j}\binom{n}{l}\binom{n-l-1}{j-l-1}j!\left\{{p \atop j}\right\}.$
By using the same method of (30) we have:
$\sum_{l=0}^{p}(-1)^l\frac{-1}{n-l}\binom{n}{l}\binom{n-l}{j-l}=-\frac{(-1)^j}{n-j}.$
Let $p=n$, then the LHS of (34) is
$\sum_{l=1}^{n-1}\sum_{j=0}^{n}(-1)^l\frac{-1}{n-l}\binom{n}{l}\binom{n-l}{j-l}j!\left\{{n \atop j}\right\}=\sum_{l=1}^{n-1}\sum_{j=0}^{n-1}(-1)^l\frac{-1}{n-j}\binom{n}{l}\binom{n-l-1}{j-l-1}j!\left\{{n \atop j}\right\}+\sum_{l=0}^{n-1}-\frac{(-1)^l}{n-l}\binom{n}{l}n!\left\{{n \atop n}\right\}.$
This is the RHS of (34) $\square$

Example 7.4. *Let $F_0, F_1, \cdots, F_n$ be the sequence of Fibonacci numbers, where $F_n=F_{n-1}+F_{n-2}$ and $\sum_{k=0}^{n}\binom{n}{k}(-1)^k F_k=-F_n$ we have:*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k k^2 F_k = -nF_{n-3}-n^2F_{n-4}. \tag{35}$$

Proof. The LHS of (35) Is:
$\sum_{k=0}^{n}(n^2\binom{n}{k}(-1)^kF_k-n(2n-1)\binom{n-1}{k}(-1)^kF_k+n(n-1)\binom{n-2}{k}(-1)^kF_k)=-n^2F_n+n(2n-1)F_{n-1}-$
$n(n-1)F_{n-2}==-nF_{n-3}-n^2F_{n-4}.$ $\square$

Example 7.5. *Let $F_0, F_1, \cdots, F_n$ be the sequence of Fibonacci numbers, where $F_n=F_{n-1}+F_{n-2}$ and $\sum_{k=0}^{n}\binom{n}{k}(-1)^k F_k=-F_n$ then:*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k k^3 F_k = -n^3F_n+n(3n^2-3n+1)F_{n-1}-3n(n-1)^2F_{n-2}+n(n-1)(n-2)F_{n-3}. \tag{36}$$

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