ROBUST DISCRETE GRID GENERATION ON PLANE IRREGULAR REGIONS.

Pablo Barrera-Sánchez[1] Guilmer F. González Flores[2]
Francisco J. Domínguez-Mota[3]

[1]U.N.A.M., Mexico City, Mexico,
pablo@athena.fciencias.unam.mx
[2]U.N.A.M., Mexico City, Mexico,
gfgf@athena.fciencias.unam.mx
[3]U.M.S.N.H., Morelia, Mexico, dmota@zeus.umich.mx

Abstract

A quite simple, robust and geometrically intuitive method for variational grid generation can be formulated in terms of separable expresions given as the sum of the values of the evaluation of a convex injective real funcion, an exponential, that is capable to distinguish in a structured grid between the convex and non-convex cells, in terms of its value on each one of the four oriented triangles defined by its corners, as it is presented in this paper.

1 Admissible grids

First of all, it is convenient to define the basic terminology and definitions used throughout this work: the planar regions where we are interested to solve the variational grid generation problem are the interior of a polygonal simple curve with positive orientation; a typical region will be denoted as Ω.
Let m and n be natural numbers greater then 2, and P be the set of points where the boundary of a region Ω is not $\mathcal{C}^1$. An $m \times n$ structured grid for such a region will be the finite set

$$G = \{P_{i,j} \mid i = 1, ..., m, j = 1, ..., n\} \qquad (1)$$

of points in the plane such that

$$P \subset G. \tag{2}$$

The grids built in this way, holding (2), are told to be admissible.
Naturally, a grid defined as in (1) is convex, if for $1 \leq i < m$ and $1 \leq j < n$ the $(m-1) \times (n-1)$ quadrilaterals with corners $P_{i,j}$, $P_{i+1,j}$, $P_{i,j+1}$ and $P_{i+1,j+1}$ respectively are, each one, convex.
The fundamental fact that allows to generate a sequence of admissible grids which converges to a convex one is the recognition that a necessary and sufficient condition for each single cell or quadrilateral to be convex is that the four triangles within it defined by its corners have positive oriented area. This has been the main key to propose several useful expressions in terms of those set of areas, which attain its minimum values when such values are positive.
So, the first basic quantity associated to a triangle $\triangle$ is twice its oriented area, which will be denoted as $\alpha(\triangle)$ or simple α for brevity.
In terms of α, another two important for a grid G over the region Ω are the minimum α

$$\alpha_{-}(G) = \min_{\triangle \in G}\{\alpha(\triangle)\}$$

and the average

$$\overline{\alpha}(G) = \frac{\text{area}(\Omega)}{4(m-1)(n-1)}$$

and it is straightforward to see that if

$$\alpha_{-}(G) > 0 \tag{3}$$

then G is convex.

2 Statement of the problem.

Let us denote as $M(\Omega)$ the set of all admissible grids for the region Ω, and

$$M_k = \{G \in M(\Omega) | \alpha_{-}(G) > k\}$$

for a real number k.
Clearly, the set of convex grids for Ω is M_0; it should be noted that for certain values of k it may happen that $M_k = \phi$.
In terms of the notation we have introduced and following the motivation of (3), the problem we want to solve can be written as

$$\max\{\alpha_{-}(G)\} \tag{4}$$

and the aim of the present work is to show that this problem can actually be solved as an unconstrained large-scale optimization problem in a very easy and robust way by means of a functional of class $\mathcal{C}^\infty$ that makes a strong use of the structural symmetries of the grids.

3 Functionals over $M(\Omega)$.

The typical expression for a functional defined over a grid G is

$$F(G) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{4} f(\triangle_{i,j}^{k})$$

which can be written in a shorter way as

$$F(G) = \sum_{q=1}^{N} f(\triangle_q) = \sum_{\triangle \in G} f(\triangle) \tag{5}$$

where $N = 4(m-1)(n-1)$ is the total number of triangles within the grid, and f denotes a function of the triangle coordinates.
Several different expressions for f have been proposed, in terms of geometrical properties we require for the grid as well as considering the motivation of the direct discretization of some continuous variational problems related to Poisson kind problems ([5], [9], [10],[11],[12]); however, convexity can only be guaranteed if the function f expresses an explicit geometrical compromise with it. Our proposal is based in the geometrical fact that such a goal can be achieved if f is capable of distinguish injectively between the triangles with positive and negative oriented area; after a convenient change of scale, this capacity will lead the functional to have its optimal values within the set M_0.
One of the simplest smooth expressions with this property is given as

$$F_t(G) = \sum_{q=1}^{N} e^{-t\alpha_q} \tag{6}$$

where t is a positive real number that will be used to scale the problem.
It is immediate to verify that the functional expression

$$\exp(-t\alpha_q)$$

indeed distinguishes the right of the wrong triangles, since in the optimization process, any increment in the value of α will cause an

increment in the value of the whole sum.
Since the order in the summation in (6) is quite irrelevant, it is natural to expect non-linear least square solutions which reflect that fact. In the next sections, it will be demonstrated that after scaling, 6 can be thought as a straight solution of problem (4).

4 Existence of optimal convex grids.

In order to present the adequate context for the solution we are looking for, let's identify the values of α of a grid G with a point in the euclidian space $\mathbf{R}^N$, where N is, as before, its total numbers of triangles. The coordinates x_i of these points will satisfy

$$\sum_{i=1}^{N} x_i = A \tag{7}$$

where A is twice the area of Ω and will represent it hereinafter.
A convex grid will be, in the same way, identified with a point in the positive "hypercuadrant". The parameter t of (6) we have already introduced will be used to change the corresponding level surfaces to be inside of that positive hypercuadrant if the set of convex grids for a region is non-empty, and such task will be possible since we know that the main component of the gradient corresponds to α_{-}. To show it, let us considerer the following lemmas, that will be presented without proof due to its simplicity.

Lemma 1 *Let u be a positive real number. There exists*

$$\mathbf{x_0} = (x_{01}, x_{02}, \cdots, x_{0N})^T \in \mathbf{R}^N$$

such that

$$F_t(\mathbf{x}_0) = \sum_{i=1}^{N} e^{-tx_{0i}} = u$$

Lemma 2 *let k be a real number, t a positive real number, and let us define*

$$\Lambda_k = \{(x_1, x_2, \cdots, x_N) \in \mathbf{R}^N \,|\, \min(x_1, x_2, \cdots x_N) > k\}.$$

Then, the level surface S_{x_0} of the functional

$$\sum_{i=1}^{N} e^{-tx_i}$$

passing thru the point

$$x_0(1, 1, \cdots, 1)^T$$

holds that

$$S_{x_0} \subset \Lambda_k$$

if the condition

$$x_0 \geq k + \frac{\log(N)}{t}$$

is satisfied.

Since (7) must be satisfied, the definition of the set Λ_k in the last lemma makes sense for the search of convex grids if the value of k is less or equal than $\frac{A}{N}$.

Lemma 3 *Let k be a real number less or equal than $\frac{A}{N}$, t a positive real number, F_t defined as in (6), Λ_k defined as in lemma 2, and P the plane*

$$P = \{(x_1, x_2, \cdots, x_N) | x_1 + x_2 + \cdots + x_N = A\}.$$

Then, for

$$\mathbf{x}_0 = (x_{01}, x_{02}, \cdots, x_{0N})^T \in \Lambda_k \cap P$$

exists a value of t such that the set

$$S_{\mathbf{x}_0} = \{\mathbf{x} = (x_1, x_2, \cdots, x_N)^T \in \mathbf{R}^N | F_t(\mathbf{x}_0) = F_t(\mathbf{x})\}$$

satisfies

$$S_{\mathbf{x}_0} \subset \Lambda_k.$$

It is important to emphasize that t can be seen as a change of parameter to force the level curves to be inside the set we require, as well as a *change of scale* of the original problem to be solved with the simpler functional

$$F(G) = \sum_{q=1}^{N} e^{-\alpha_q}. \tag{8}$$

Next, it is easy to show the following results.

Theorem 1 *Let $k < \frac{A}{N}$, and let us suppose that there exists $\mathbf{x}_0 \in M_0$. Then, for a value of t large enough the optimization problem*

$$\min\{F_t(G) | G \in M_k\}$$

has a solution $\hat{G} \in M_0$.

Demonstration:
Let us denote, for the real number a such that $a \geq Ne^{-\frac{tA}{N}}$,

$$L_a = \{\mathbf{x} \in P | a \geq F_t(x)\}$$

where

$$P = \{(x_1, x_2, \cdots, x_N) | x_1 + x_2 + \cdots + x_N = A\},$$

and

$$T_0 = \Lambda_0 \cap P,$$

with

$$\Lambda_0 = \{(x_1, x_2, \cdots, x_N) \in \mathbf{R}^N | \min(x_1, x_2, \cdots x_N) > 0\}.$$

Let t be such that the level surface

$$S_{G_0} = \{G \in \mathbf{R}^N | F_t(G_0) = F_t(G)\}$$

satisfies

$$S_{G_0} \subset \Lambda_0.$$

The intersection

$$S_{G_0} \cap P$$

is non-empty since $\mathbf{x}_0 \in P$, and then

$$S_{G_0} \cap P \subset \Lambda_0 \cap P = T_0$$

this is, the level surface of the problem restricted to P is completely contained in T_0.
It is easy to verify that

$$S_{G_0} \cap P = \partial(L_{F_t(G_0)})$$

where ∂ denotes the boundary of the set, and from here

$$\forall G \in M_k \setminus L_{F_t(G_0)}$$

we have

$$F_t(G) > F_t(G_0)$$

and

$$\forall \hat{G} \in M_k \setminus L_{F_t(G_0)}$$
$$F_t(G) \leq F_t(G_0).$$

Since

$$G_O \in L_{F_t(G_0)}$$

then we have

$$G_O \in L_{F_t(G_0)} \cap M_k \neq \phi.$$

Finally, $L_{F_t(G_0)} \cap M_k$ is a closed subset of $L_{F_t(G_0)}$ due to the continuity of F_t, and the conclusion of the theorem follows immediately.□

Theorem 2 *Let Ω be a polygonal region of positive oriented area A, and let be, as before, $M(\Omega)$ its set of admissible grids and*

$$M_k = \{G \in M(\Omega) | \alpha_-(G) > k\}$$

for a real k real that satisfies

$$k < \frac{A}{N}.$$

If exists $G_0 \in M_0(\Omega)$, then the optimization problem

$$\min\{F_t(G) | G \in M_k\}$$

has a solution $\hat{G} \in M_0$ when

$$t > \frac{\log N}{\alpha_-(G_0)}.$$

Demonstration:
Lets denote the coordinates of G_0 as

$$G_0 = (x_{01}, x_{02}, \cdots, x_{0N})^T$$

and so we can write

$$F_t(G_0) = \sum_{i=1}^{N} e^{-tx_{0i}}$$

and

$$\alpha_-(G_0) = \min_{1 \leq i \leq N} \{x_{0i}\}.$$

From 1 y 2, we know that the level surface of G_0 is contained inside $\Lambda_{k'}$ if

$$k' \leq \frac{\log F_t(G_0)}{t}.$$

Since the function

$$\psi(t) = -\frac{\log F_t(G_0)}{t}$$

is increasing because G_0 is convex 3, it is sufficient to guarantee $\psi(t)$ to be positive, which is equivalent to show that there exists a value of t such that

$$-\log F_t(G_0) > 0$$

this is

$$F_t(G_0) < 1. \tag{9}$$

To satisfy the last expression, it is just needed to note that

$$\alpha_{-}(G_0) \leq x_{0i}, \quad 1 \leq i \leq N$$

and from here

$$F_t(G_0) \leq Ne^{-t\alpha_{-}(G_0)}$$

and in consequence, a sufficient condition to guarantee 9 is

$$Ne^{-t\alpha_{-}(G_0)} < 1$$

which can be written as

$$t > \frac{\log N}{\alpha_{-}(G_0)} \tag{10}$$

as we wanted.□

It seems necessary to know the components of a convex grid to estimate 10. However, it is not, since we can previously map our region by

$$(x_1, x_2, \cdots, x_N)^T \quad \mapsto \quad \frac{1}{\sqrt{\overline{\alpha}(G_0)}}(x_1, x_2, \cdots, x_N)^T \tag{11}$$

where, as before,

$$\overline{\alpha}(G_0) = \frac{\text{area}(\Omega)}{N}$$

and this scaled problem will hold

$$\overline{\alpha}(G_0) = 1 \tag{12}$$

in such a way that since

$$\alpha_{-}(G_0) \leq \overline{\alpha}(G_0)$$

we will have for 10

$$t > \frac{\log N}{\alpha_{-}(G_0)} \geq \log N.$$

This is a bound that allows an automatic implementation of the problem in terms of the dimension of the problem, and since log is a function that in creases slowly enough, it will not represent any major problem in the practical implementation for several standard dimensions. In table 1 some typical values are shown for grids with the same number of horizontal and vertical nodes in the boundary; m stands for that number of points and $N = 4(m-1)^2$ is the corresponding associated number of triangles.

m	N	$\log N$
5	64	4.1589
10	324	5.7807
15	784	6.6644
20	1444	7.2752
25	2304	7.7424
30	3364	8.1209
35	4624	8.4390
40	6084	8.7134
45	7744	8.9547
50	9604	9.1699

Table 1: Values of $\log N$.

5 Robustness of F_t.

The parameter k in theorem 3 is almost only required to stablish the definition domain of the optimization problem, to assure that the set M_k is a non-empty set, but this task is done by assuming the existence of a convex admissible grid; under such hypothesis, many possible negative values of k can be used.
What this means is that due to the smoothness of the functional 6, which is defined over the totality of the euclidian space $\mathbf{R}^N$, *our initial grid can be any admissible grid*, generated for instance for transfinite interpolation, and taking k as its value of α_-, once that the mapping 11 has been done to simplify the computational work required, *id est* our proposed method is a robust grid generator.
Its easiness as well as its geometrical compromise with convexity turns out to be quite evident, and furthermore: it is reflected in the practical implementation, and as we will see in the next section, in the possibility of using 6 as a barrier to avoid other functionals to provoke non-convexity when we used it in convex linear combinations.

6 Combination of functionals.

As discussed in [5], a natural straight way to generate grids that show several different geometrical properties, as smoothness and orthogonality, is achieved by taking convex linear combinations between the

functionals that provide them. However, in the general case, convexity will not be guaranteed if none of the functionals used has its optimal values in convex grids.
The two main requirements for the grids in order to solve differential kind problems with them are smoothness and orthogonality, but unfortunately, the corresponding discrete functionals, length

$$L(G) = \sum_{q=1}^{N}(\|\mathbf{a}_q\|^2 + \|\mathbf{b}_q\|^2) = \sum_{q=1}^{N} l(\triangle_q) \tag{13}$$

and orthogonality

$$O(G) = \sum_{q=1}^{N}(\mathbf{a}_q^T\mathbf{b}_q)^2 = \sum_{q=1}^{N} o(\triangle_q) \tag{14}$$

where **a** and **b** are two column vectors representing two sides of a triangle in our formulation and l and o represent the functional value of length and orthogonality respectively calculated on it, can be minimized by non-convex grids.
To prove that a convex linear combination of 13 or 14 with 6 will also have its optimal values within the set of the admissible convex grids, we will considerer another couple of elementary results.

Lemma 4 *Let P be the plane*

$$P = \{(x_1, x_2, \cdots, x_N) | x_1 + x_2 + \cdots + x_N = A\},$$

t a positive real numbers and a a real number such that $a \geq Ne^{-\frac{tA}{N}}$. The level surfaces

$$L_a = \{\mathbf{x} \in P | a \geq F_t(x)\}$$

satisfy:

1. *They are non-empty close sets,*
2. *If $a < b$, then $L_a \subset L_b$,*
3. *$\forall \mathbf{x} \in P \setminus L_a$, $F_t(\mathbf{x}) > a$,*
4. *The boundary of L_a is the set*

$$\{\mathbf{x} \in P | F_t(x) = a\},$$

5. *They are convex sets, since if*

$$\mathbf{x}_1 = (x_{11}, \cdots, x_{1N})^T, \mathbf{x}_2 = (x_{21}, \cdots, x_{2N})^T \in L_a,$$

then for $0 \leq \lambda \leq 1$ *y* $1 \leq i \leq N$ *we have*

$$e^{-t(\lambda x_{1i}+(1-\lambda)x_{2i})} < \lambda e^{-tx_{1i}} + (1-\lambda)e^{-tx_{2i}}$$

which implies that

$$F_t(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) < \lambda F_t(\mathbf{x}_1) + (1-\lambda)F_t(\mathbf{x}_2) \leq a,$$

6. *They are bounded set (and in consequence compacts by 1.), since must be satisfied and for* $1 \leq i \leq N$

$$e^{-tx_i} \leq F_t(\mathbf{x}) \leq a,$$

and therefore they are a family of topological $(N-1)$ *spheres.*

Theorem 3 *Let* m *be a non-negative real number, and* L_a *is in lemma 4. Then*

1. $L_{a-m} \subset L_a$,
2. *if* $a - m < Ne^{-\frac{tA}{N}}$, *then*

$$L_{a-m} = \phi,$$

Demonstration:
It is straightforward from

$$\sum_{i=1}^{N} e^{-tx_i} + m = a.\Box$$

And finally, from the lemmas 3 and 2, the theorems 4 and 3 and the fact that the functionals 13 and 14 are non-negative, we have the next

Theorem 4 *Let* σ *be a real number such that* $0 \leq \sigma \leq 1$. *If there exists a convex admissible grid for a region* Ω, *then each one of the functionals*

$$\sigma F_t(G) + (1-\sigma)L(G) = \sigma \sum_{q=1}^{N} \{e^{-t\alpha_q} + (1-\sigma)l(\triangle_q)\}$$

$$\sigma F_t(G) + (1-\sigma)O(G) = \sigma \sum_{q=1}^{N} \{e^{-t\alpha_q} + (1-\sigma)o(\triangle_q)\}$$

has a solution $\hat{G} \in M_0$ *for a value of* t *large enough.*

Again, t large enough has the same meaning that in theorem 4.
The main difficult in the implementation of these combinations yields on the fact that the magnitude orders of the different functionals can be very different, so they must be previously normalized with respect to the values for a standard grid. The typical normalization constants for orthogonality and length are given as

$$\frac{1}{N\overline{\alpha}^2}$$

end

$$\frac{1}{2N\overline{\alpha}}$$

respectively, and for the F_t functional we used

$$\frac{1}{\exp(10.0)}$$

since the upper bound for the product $-tx_i$ was estimated as 10.0 following the values of the table 1.
Some of the grids we obtained are presented in the next section.

7 Some results.

Three boundaries with a high degree of complexity were selected: Great Britain, Russia and Mexico. Each one was approximated with 41 points by side, and in the case of Mexico, the corners were taken as the geographical limits of the country, the other two choices for the corners were arbitrary. The initial grids were generated by interpolation, next scaled to satisfy 12; the convex grids were generated with a value of $t = 10.0$ and $\sigma = 0.5$.
The respective figures are 1 and 2 for Russia, 3 and 3 for Great Britain and 5 and 6 for Mexico.

8 Conclusions.

From elementary considerations, and involving the use of the most classical quantities used in variational grid generation, namely area, orthogonality and length, it was indeed possible to develop a very robust method with simple geometrical requirements, thru an expression of amazing simplicity as (6).

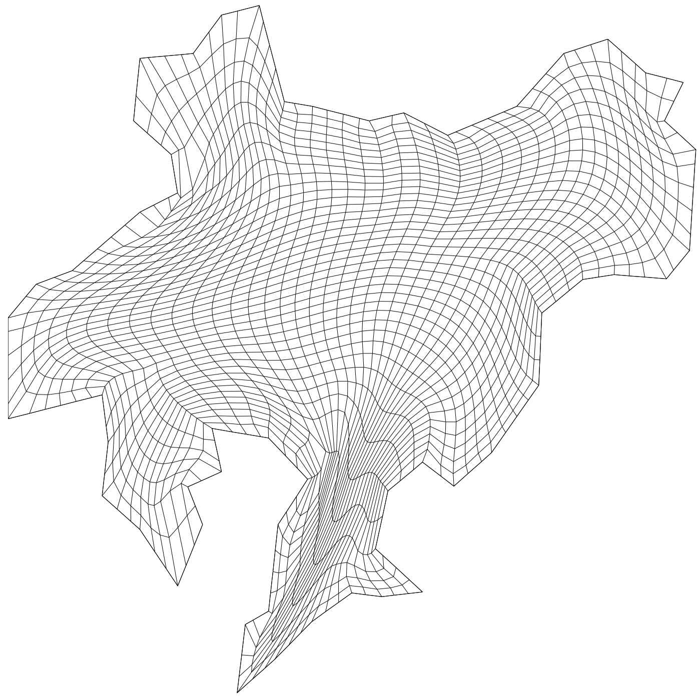

Figure 1: Grid for Russia, $F_t - L$.

Further analysis and other alternative methods must be developed and studied, trying now to improve the performance that can be lost due to the computational difficulties in the calculations regarding exponential functions, looking for some other convex bounded expressions that can also injectively distinguish the orientation within the grids as it was done in this work.

Note: A free software for structured grid generation, that includes this method and other very effective variational approaches is available on the World Wide Web at the site

http://tycho.fciencias.unam.mx/˜unamalla .

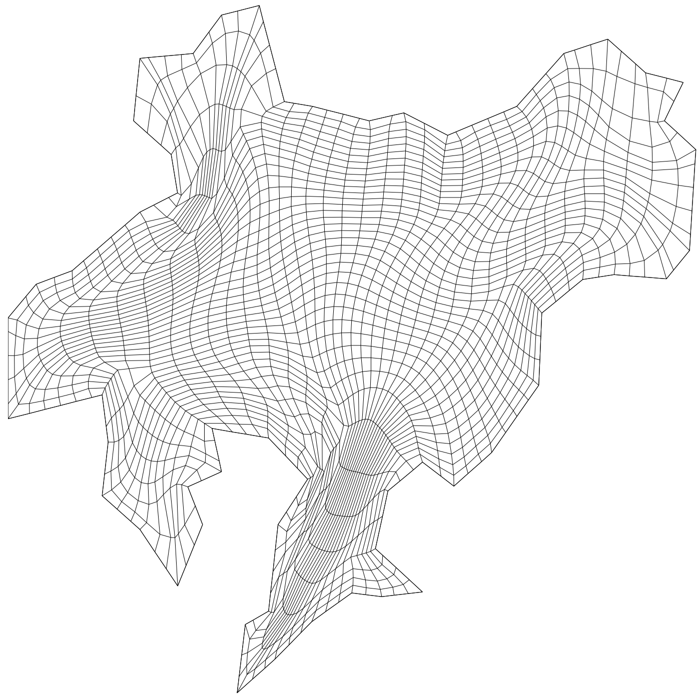

Figure 2: Grid for Russia, $F_t - O$.

References

[1] J. E. Castillo. *Mathematical aspects of numerical grid generation.*,SIAM Press, 1991.

[2] B.K. Soni, J.F Thompson and N.P. Weatherhill. *Handbook of Grid Generation*, CRC Press, 1999.

[3] Thompson J. F., Z. U. A. Warsi and C. Wayne Mastin. *Numerical Grid Generation: Foundations and Applications*, Elsevier Science Publications Co., Inc. New York N.Y., 1985.

[4] S. A. Ivanenko. *Curvilinear Grids of Convex Quadrilaterals*, J. Comp. Physics, 1988.

[5] P. Knupp and S. Steinberg. *Fundamentals of Grid Generation*, CRC Press, 1992.

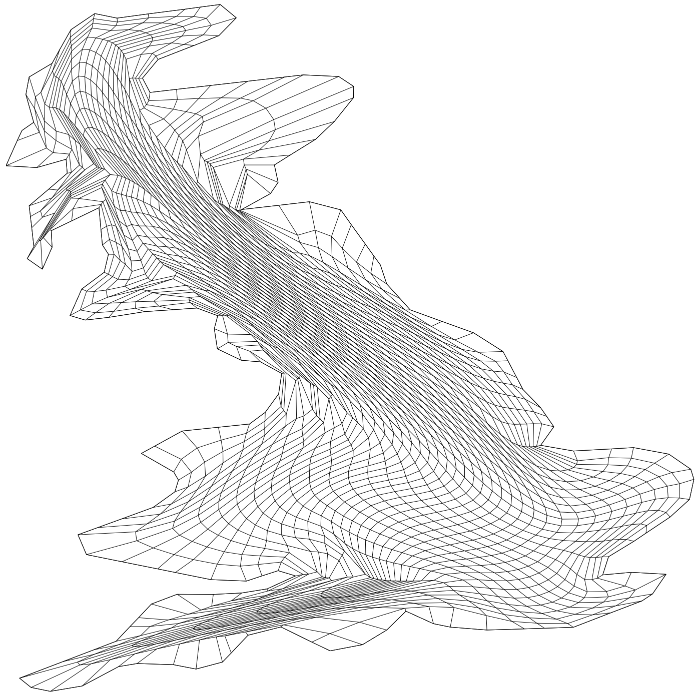

Figure 3: Grid for Great Britain, $F_t - L$.

[6] Hardy, Littlewood and Pólya. *Inequalities*, Cambridge Mathematical Library, 1934.

[7] F.J. Domínguez-Mota. *Sobre la generación de mallas cuasi-ortogonales en el plano (in spanish)*, Ph.D. Thesis (*draft*),U.N.A.M., 2001.

[8] P. Barrera S., Guilmer F. González F. and F.J. Domínguez-Mota *Some experiences on orthogonal grid generation*,Proceedings of the 9^{th} International Meshing Roundtable, 2000.

[9] P. Barrera S., Guilmer F. González F. and F.J. Domínguez-Mota *Variational grid generation using exponentials of area*, Proceedings of the 10^{th} International Meshing Roundtable, 2001.

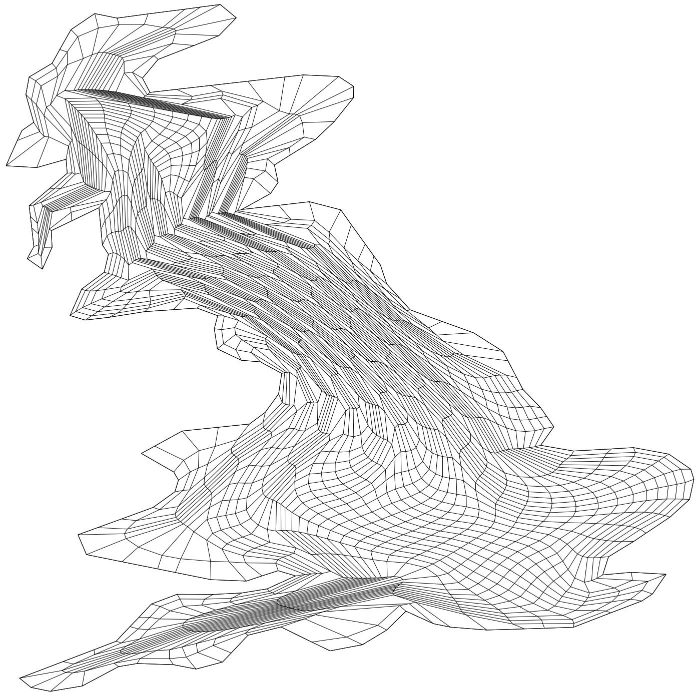

Figure 4: Grid for Great Britain, $F_t - O$.

[10] J. G. TINOCO R. *Funcionales discretos para la generacion de mallas suaves y convexas sobre regiones planas irregulares (in spanish)*,Ph.D. Thesis,C.I.M.A.T., 1997.

[11] J. G. TINOCO R. AND P. BARRERA S. *Smooth and Convex Grid Generation over General Plane Regions*, Mathematics and Computer in Simulation, 1997.

[12] J. G. TINOCO R. AND P. BARRERA S. *Area Functionals in Plane Grid Generation*, 6^{th} Conference in Numerical Grid Generation in Computational Field Simulation Conference, London, 1998.

[13] P. BARRERA S. AND A. DOMÍNGUEZ. *Curvilinear coordinate systems for planar grid generation*,U.N.A.M. internal reports, 1992.

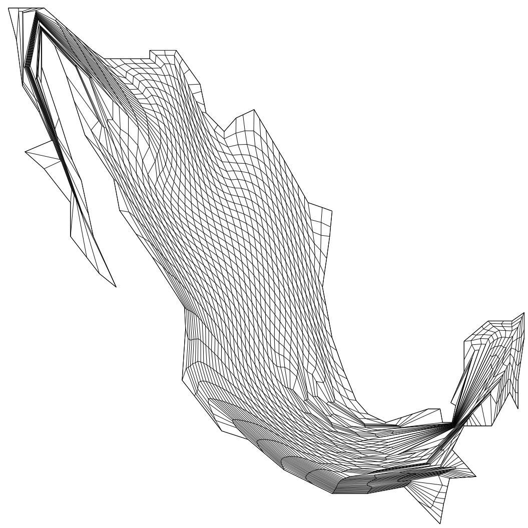

Figure 5: Grid for Mexico, $F_t - L$.

[14] UNAMALLA (SOFTWARE). A system for 2D grid generation, *free distribution available at* `http://tycho.fciencias.unam.mx/\~unamalla`

[15] C.D. Mobley y R.J. Stewart. *2D Orthogonal Grid Generation*, Journal of Computational Physics 34, 1980.

[16] I. Schur. *Uber eine Klasse von Mittelbildungen mit Anwendungen auf die Determinantentheorie*, Sitzunsber. Berlin. Math., 1923.

[17] A. Wayne Roberts and Dale E. Varberg. *Convex Functions*, Academic Press, 1973.

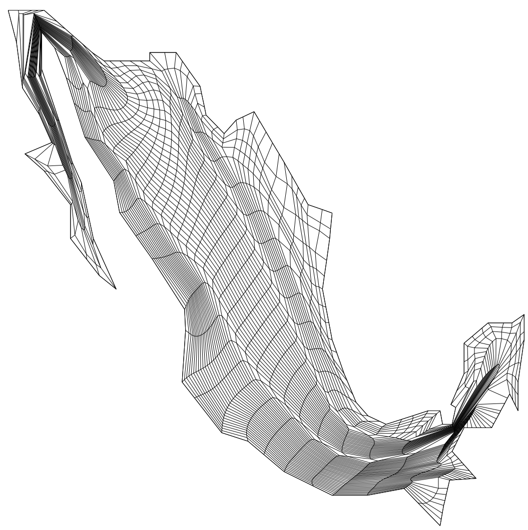

Figure 6: Grid for Mexico, $F_t - O$.