SOME PROPERTIES OF AREA FUNCTIONALS IN NUMERICAL GRID GENERATION

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ABSTRACT

In this paper we present some general properties concerning area functionals, both continuous and discrete, used in the variational approach of numerical grid generation. In particular we introduce a new proof of a theorem concerning the classical area functional which has served us as a guide in the design of new area functionals yielding better performance than the currently used. In particular these new functionals produce grids with a better control in the areas of the cells.

Keywords: area, discrete functional, continuous functional, variational grid generation

INTRODUCTION

A grid or mesh $\mathbf{x}(\xi,\eta) = (x(\xi,\eta), y(\xi,\eta))^t$ over a region $\Omega \subset R^2$ is a continuous function

$$\mathbf{x} : U_2 \to \Omega$$

where U_2 is the unit square $[0,1] \times [0,1]$.

We can see that if we choose a coordinate line $\xi = cte$ or $\eta = cte$ in U_2, its image will be a curve. These curves in Ω are called in a natural manner, coordinate curves and the intersection points are called knots. In a sense, a grid is a change of coordinates. If we take a net in U_2; that is, a collection of coordinate lines in U_2, function $\mathbf{x}$ gives us a net in Ω. This justifies the term grid or mesh. In fact, a numerically generated grid is just a collection of points that are approximate images of the knots in U_2 under the continuous grid.

We restrict our attention to grids such that

$$\mathbf{x}(\partial U_2) = \partial\Omega$$

These are said to conform the boundary. There could exist many grids on Ω. We are interested in those with certain properties:

1. In the first place, we want that different points in U_2 have as images different points in Ω; it can be seen that this implies that two different coordinate lines without intersection in U_2, will have images without intersection in Ω. In other words $\mathbf{x}$ must be 1-1.

2. We also want that $\mathbf{x}$ fills Ω; that is, $\mathbf{x}$ is to be an onto mapping.

3. A desirable property is that the image of a smooth curve in U_2 be a smooth curve in Ω; that is, we want functions $x(\xi,\eta)$, $y(\xi,\eta)$ to have continuous derivatives when considered as functions of ξ and η separately.

A function having these three properties is called a diffeomorphism between U_2 and Ω. We state grid generation problem as:

Given a diffeomorphism

$$\mathbf{x} : \partial(U_2) \to \partial\Omega$$

to extend it to a diffeomorphism

$$\mathbf{x} : U_2 \to \Omega.$$

The material in the present work is organized as follows: in section 1., we explain briefly the main ideas of the continuous version of variational grid generation. The most classical functional are reviewed and its properties are listed, with special, but nevertheless brief emphasis on the classical Area functional. At the end of the section we introduce the notion of general Area Functionals and state and prove a property shared by all of them. This is one of the main objectives. In section 2., the discrete approach is introduced; like in section 1., the classical functionals and its properties are presented. Again, the Classical Discrete Area Functional will deserve special attention, its main properties are stated. Later in the same section, the general form of Discrete Area Functionals is introduced and it is seen that, under certain conditions, all of them have the same key property that makes it useful for grid generation. Finally, in section 3., some special area functionals are shown; these functional have been developed using the ideas of section 2., and have been used extensively for generating convex grids. Some graphs of grids generated with them are shown.

1. CONTINUOUS VARIATIONAL GENERATION

In this section we outline in a brief manner the main ideas concerning continuous variational grid generation methods. We will introduce just the basic elements in order to discuss the most used of the so called classical continuous methods. Our approach is mainly due to [9, 12] where a wider discussion can be found.

The notion of functional is quite ample; however, we just use a special kind of it in this work. We say that a *continuous grid generation functional* is a quantity of the form

$$I(\mathbf{x}) = \int_0^1 \int_0^1 L(x_\xi, x_\eta, y_\xi, y_\eta) d\eta d\xi.$$

associating with every mesh $\mathbf{x} : U_2 \to \Omega$ a real number. L is a function containing information about certain geometric quantities to be controlled in order to generate adequate grids on Ω, with the property of being boundary conforming.

The problem of continuous variational grid generation is to find $\mathbf{x}^*$, boundary conforming and making the value $I(\mathbf{x})$ a minimum. That is, it is required to calculate $\mathbf{x}^*$ such that $I(\mathbf{x}^*) = \min_{\mathbf{x}} I(\mathbf{x})$.

From Variational Calculus it follows that $\mathbf{x}^*$ must be a solution of the boundary value problem

$$\begin{array}{rcll} \dfrac{d}{d\xi}\left(\dfrac{\partial L}{\partial x_\xi}\right) + \dfrac{d}{d\eta}\left(\dfrac{\partial L}{\partial x_\eta}\right) & = & 0 & \\ & & & \mathbf{x}(\partial U_2) \textit{ given} \\ \dfrac{d}{d\xi}\left(\dfrac{\partial L}{\partial y_\xi}\right) + \dfrac{d}{d\eta}\left(\dfrac{\partial L}{\partial y_\eta}\right) & = & 0 & . \end{array}$$

This is a partial differential equations system -very often coupled and non linear- with boundary conditions given by the value of the grid on the contour. These conditions are called *Euler-Lagrange Equations* and we will refer to them simply as*E-L Equations*.

The common practice is to solve this boundary value problem using some finite-difference scheme to discretize the differential operators involved; then some iterative method to get the solution of the resulting system of algebraic equations. The initial guess in this iterative process is generated using some algebraic method as TFI, by example ([9]).

Perhaps the most widely used continuous functionals are the *Length, Orthogonality, and Area*. The last one deserves special attention in this work. We present briefly the main features of them; a detailed discussion can be found in ([9, 13]).

1.1 The Length Functional

The classical length functional Φ_l in its simplest form is obtained with

$$L(x_\xi, x_\eta, y_\xi, y_\eta) = \lambda = x_\xi^2 + x_\eta^2 + y_\xi^2 + y_\eta^2$$

By calculating the minimal value of the functional

$$\Phi_l(\mathbf{x}) = \int_0^1 \int_0^1 \lambda d\eta d\xi$$

we try to reach a grid with the characteristic that all the lengths of the coordinate lines be the closest possible to a given, constant value. Moreover, it is easy to see that the ideal minimum of $\Phi_l(\mathbf{x})$ is obtained in a mapping satisfying the Cauchy-Riemann Equations

$$x_\xi = y_\eta \qquad x_\eta = -y_\xi$$

in a least-squares sense. In fact the Euler- Lagrange Equations for Φ_l are

$$\begin{aligned} x_{\xi\xi} + x_{\eta\eta} &= 0 \\ y_{\xi\xi} + y_{\eta\eta} &= 0 \end{aligned}$$

just Laplace Equations for each of the components of the mapping.

In this case the E-L Equations are linear and uncoupled, the simplest case to consider. These form what is known as an Elliptic Partial Differential Equations System. Despite of its simplicity, this functional does not lead to satisfactory grids because for almost non-convex regions the optimal grids are folded; that is they are not 1-1. Some examples of grids generated using the length functional can be found in [14].

1.2 The Area Functional

In its simplest version, the continuous area functional Φ_A tries to produce grids such that variation of the jacobian be the smallest possible. It has the form

$$\begin{aligned} \Phi_A(\mathbf{x}) &= \int_0^1 \int_0^1 (x_\xi y_\eta - x_\eta y_\xi)^2 d\eta d\xi \\ &= \int_0^1 \int_0^1 J^2 d\eta d\xi \end{aligned}$$

where $J = x_\xi y_\eta - x_\eta y_\xi$.

The Euler - Lagrange equations for this functional are more complicated than the corresponding to the length functional:

$$\begin{aligned} y_\eta^2 x_{\xi\xi} - x_\eta y_\eta y_{\xi\xi} - 2y_\xi y_\eta x_{\xi\eta} + \\ +(x_\xi y_\eta + x_\eta y_\xi) y_{\xi\eta} + y_\xi^2 x_{\eta\eta} - x_\xi y_\eta y_{\eta\eta} &= 0 \\ -x_\eta y_\eta x_{\xi\xi} + x_\eta^2 y_{\xi\xi} + (x_\xi y_\eta + x_\eta y_\xi) x_{\xi\eta} - \\ -2x_\xi x_\eta y_{\xi\eta} - x_\xi y_\xi x_{\eta\eta} + x_\xi^2 y_{\eta\eta} &= 0 \end{aligned}$$

In fact, they are a non-elliptic, coupled and quasilinear system. Until now, there are not positive results in the direction of existence nor uniqueness of solutions to it. Grids produced with this functional are not smooth and, except in simple cases, folded. One of the goals of the present work is to show that these equations are essentially valid for any area functional.

1.3 The Orthogonality Functional

This functional is used trying to reach grids with orthogonal coordinate lines. This is made by requiring that tangents to these coordinate lines to be orthogonal at every point, condition that can be described by the relation $O = \mathbf{x}_\xi^t \mathbf{x}_\eta^t = 0$. A grid with this characteristic does not exist for most regions; instead, we try to calculate a mesh as near as possible to orthogonality. One way to do this is to calculate a grid satisfying the former condition in a least-squares sense; that is, the minimum of

$$\begin{aligned} \Phi_O(\mathbf{x}) &= \int_0^1 \int_0^1 (\mathbf{x}_\xi^t \mathbf{x}_\eta^t)^2 d\eta d\xi \\ &= \int_0^1 \int_0^1 (x_\xi x_\eta + y_\xi y\eta)^2 d\eta d\xi \end{aligned}$$

is calculated. E-L equations for this functional are

$$\begin{aligned} x_\eta^2 x_{\xi\xi} + x_\eta y_\eta y_{\xi\xi} + (4x_\xi x_\eta + 2y_\xi y_\eta) x_{\xi\eta} + \\ +(x_\xi y_\eta + x_\eta y_\xi) y_{\xi\eta} + x_\xi^2 x_{\eta\eta} + x_\xi y_\xi y_{\eta\eta} &= 0 \\ x_\eta y_\eta x_{\xi\xi} + y_\eta^2 y_{\xi\xi} + (x_\xi y_\eta + x_\eta y_\xi) x_{\xi\eta} + \\ +(4y_\xi y_\eta + 2x_\xi x_\eta) y_{\xi\eta} + x_\xi y_\xi x_{\eta\eta} + y_\xi^2 y_{\eta\eta} &= 0 \end{aligned}$$

again these are quasilinear, coupled and not elliptic. Very often the solving process does not converge; when it does, solution grids are extremely folded and because of this, it is urged not to use this functional alone.

1.4 Combination of Functionals

Knupp and Steinberg ([9, 10]) introduce linear combinations of the three previous functionals with the aim of reaching grids sharing, at certain extent, the characteristics offered by each functional when used separately. The general form of this combined functional is

$$\Phi(\mathbf{x}) = \sigma_l \Phi_l(\mathbf{x}) + \sigma_A \Phi_A(\mathbf{x}) + \sigma_O \Phi_O(\mathbf{x})$$

where $\sigma_l, \sigma_A, \sigma_O \geq 0$.

The choice of parameters σ_l, σ_A and σ_O depends strongly on personal experience and this is a serious drawback for using this combination. Combinations with $\sigma_O = 0$ and small σ_l , produce to the so-called area-length functionals; these are of common use because for non-convex regions, geometrically not too complicated, their minimization produces unfolded and rather smooth grids; however for most of non simple regions, convexity (unfolding) is not reached.

An automatic generator is obtained putting $\sigma_l = 0$ $\sigma_A = 1/2$ $\sigma_O = 1/2$ giving rise to the so-called Knupp's Area - Orthogonality Functional. The corresponding E-L equations are

$$\begin{array}{rcl} (x_\eta^2 + y_\eta^2)x_{\xi\xi} + 4x_\xi x_\eta x_{\xi\eta} + & & \\ +2(x_\xi y_\eta + x_\eta y_\xi)y_{\xi\eta} + (x_\xi^2 + y_\xi^2)x_{\eta\eta} & = & 0 \\ (x_\eta^2 + y_\eta^2)y_{\xi\xi} + 2(x_\xi y_\eta + x_\eta y_\xi)x_{\xi\eta} + & & \\ +4y_\xi y_\eta y_{\xi\eta} + (x_\xi^2 + y_\xi^2)y_{\eta\eta} & = & 0 \end{array}$$

Grids generated by the area-orthogonality functionals have small variation of the jacobian, are smooth, and they are unfolded if region is relatively simple.

1.5 General Continuous Area Functionals

A continuous area functional will be an expression of the form

$$\int_0^1 \int_0^1 L(J) d\eta d\xi.$$

In order to simplify the expression for the corresponding Euler-Lagrange equations, let us agree in denoting $\frac{dL}{dJ} = L'$ and $\frac{d^2 L}{dJ^2} = L''$. Then we have

$$\begin{array}{rcl} \frac{d}{d\xi}\left(\frac{\partial L}{\partial x_\xi}\right) & = & \frac{d}{d\xi}\left(L' \frac{\partial J}{\partial x_\xi}\right) \\ & = & \frac{d}{d\xi}\left(y_\eta L'\right) \\ & = & y_{\xi\eta} L' + y_\eta L'' \frac{dJ}{d\xi} \\ \frac{d}{d\eta}\left(\frac{\partial L}{\partial x_\eta}\right) & = & \frac{d}{d\eta}\left(L' \frac{\partial J}{\partial x_\eta}\right) \\ & = & -\frac{d}{d\eta}\left(y_\xi L'\right) \\ & = & -y_{\xi\eta} L' - y_\xi L'' \frac{dJ}{d\eta} \\ \frac{d}{d\xi}\left(\frac{\partial L}{\partial y_\xi}\right) & = & \frac{d}{d\xi}\left(L' \frac{\partial J}{\partial y_\xi}\right) \\ & = & -\frac{d}{d\xi}\left(x_\eta L'\right) \\ & = & -x_{\xi\eta} L' - x_\eta L'' \frac{dJ}{d\xi} \\ \frac{d}{d\eta}\left(\frac{\partial L}{\partial y_\eta}\right) & = & \frac{d}{d\eta}\left(L' \frac{\partial J}{\partial y_\eta}\right) \\ & = & \frac{d}{d\eta}\left(x_\xi L'\right) \\ & = & x_{\xi\eta} L' + x_\xi L'' \frac{dJ}{d\eta} \end{array}$$

Thus the Euler-Lagrange equations can be written as

$$\begin{array}{rcl} L''\left[x_\eta \frac{dJ}{d\xi} - x_\xi \frac{dJ}{d\eta}\right] & = & 0 \\ L''\left[y_\eta \frac{dJ}{d\xi} - y_\xi \frac{dJ}{d\eta}\right] & = & 0 \end{array}$$

From this we conclude that the Euler-Lagrange equations are the same for all continuous area functionals, up to the factor L''.

For the continuous version of the classic area functional $L'' = 2$. Therefore a grid satisfying equations for the classic functional, also satisfies the equations corresponding to any other area functional. Conversely, a grid satisfying the Euler-Lagrange equations for an area functional and such that the factor L'' doesn't vanish, also satisfies the equations for the classic area functional.

This result, which theoretically could not be important, transforms into a fact of paramount importance if we recall that the Euler-Lagrange equations are only approximately solved using finite differences schemes, and in this context the correct choice of L'' could improve the quality of the numerical solution.

Algebraic manipulations of the system of Euler-Lagrange equations, leads to write them as

$$L'' \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{dJ}{d\xi} \\ \frac{dJ}{d\eta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and if we considerer only non-singular grids; that is, those such that

$$\begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix}$$

is non-singular, we can infer that for such grids E-L equations are equivalent to

$$L'' \begin{pmatrix} \frac{dJ}{d\xi} \\ \frac{dJ}{d\eta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and from these it follows that if $L'' \neq 0$, J is constant. This constant must be the value of the area of region Ω. If there exists a solution, this shares the property we were looking for. Again, the choice of L can be crucial to get a good numerical approximation to a solution.

2. DISCRETE VARIATIONAL GRID GENERATION

In the discrete variational approach, a function of the inner points is designed and its minimum is expected to be attained in a grid with good geometrical properties; a large scale optimization algorithm is used. This kind of functions are called *discrete functionals*. The first studies using this approach were done by Castillo and Steinberg ([4, 5, 6]), who introduced the Length and Area functionals. Several works have been developed in this topic, among them we can mention Barrera, Pérez and Castellanos ([1, 2, 3]). The methods developed by them failed to obtain convexity when the region was not simple. Tinoco and Barrera ([14, 15, 16]introduced a new family of functionals, named *Adaptive Smoothness Functionals*, whose optimization produces smooth and convex grids over general plane regions, regardless of whether the initial grid is convex or not. The authors introduced a procedure to reduce an effect that is inherent to the smoothness functionals, namely the appearance of some cells with large areas. Theoretical results on the effectiveness of this method were presented.

2.1 Notation and Definitions

Let Ω be a plane region. We say that Ω is a polygonal region if its boundary ($\partial\Omega$) is a not self-intersecting closed polygon. In this paper only polygonal regions will be considered, and the term *region* will mean polygonal region. Let m and n be positive integer numbers, and let Ω be a region.

Definition 1. An $m \times n$ grid G over Ω, $\{P_{i,j}, \quad i = 1, 2, \cdots, m \quad j = 1, 2, \cdots, n\}$ is a set of points with the following property:

$$\begin{aligned} \partial\Omega = \quad & Polygon\{P_{1,1}, \cdots, P_{1,n}, P_{2,n}, \cdots, P_{m,n}, \\ & P_{m,n-1}, \cdots, P_{m,1}, P_{m-1,1}, \cdots, P_{1,1}\} \end{aligned}$$

that is, $\partial\Omega$ is the boundary of the grid $\{P_{i,j}\}$.

The i, j cell of the grid, denoted by $C_{i,j}$, is the quadrilateral whose vertices are $P_{i,j}, P_{i+1,j}, P_{i+1,j+1}, P_{i,j+1}$. An $m \times n$ grid determines $(m-1)(n-1)$ cells. The diagonals of the i, j cell, determine four triangles $\Delta^{(1)}_{i,j}, \Delta^{(2)}_{i,j}, \Delta^{(3)}_{i,j}, \Delta^{(4)}_{i,j}$, so the total number of triangles determined by the grid cells is $N = 4(m-1)(n-1)$.

The sets $\{P_{1,j}, P_{2,j}, \cdots, P_{m,j} \quad j = 1, 2, \cdots, n\}$ will be called *horizontal coordinate lines*; while the sets $\{P_{i,1}, P_{i,2}, \cdots, P_{i,n} \quad i = 1, 2, \cdots, m\}$ will be called *vertical coordinate lines*. When explicit mention of a triangle Δcab is needed, it means that the segments ca and ab are on a coordinate line and that cb is a diagonal of the corresponding cell. With this convention, the following quantities are defined for a triangle in the grid:

$l(\Delta cab) = ||c-a||^2 + ||b-a||^2$,

$o(\Delta cab) = (c-a)^t(b-a)$,

$\alpha(\Delta cab) = (b-a)^t J_2 (c-a)$ where

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\bar{\alpha} = \frac{1}{N}\sum \alpha(\Delta)$, summation is made over all the N triangles in the grid.

Observe that l is the sum of the squares of the lengths of the triangle sides that are on coordinate lines, o is the inner product of these sides and α is twice the oriented area of the triangle, and $\bar{\alpha}$ is the average area of the triangles.

A grid is called *convex* if each of its cells is a convex quadrilateral. It is not difficult to see that the cell $C_{i,j}$ will be convex if, and only if, the four triangles in it have a positive area.

As we have said one of the main goals in grid generation is to calculate a convex grid over a given plane region, given that points on the boundary are fixed. Other desirable features for a grid are smoothness and small cell area variation.

The functionals to be considered here are of a rather special form, namely

$$F(G) = \sum_{i=1}^{m-1}\sum_{j=1}^{n-1}\sum_{p=1}^{4} f(\Delta_{i,j}^{(p)})$$

if an order is assigned to the triangles in the grid; F can be written simply as

$$F(G) = \sum_{q=1}^{N} f(\Delta_q)$$

where the choice of f depends on the desired properties of the optimal grid. Among all the grids of dimension $m \times n$ with the same boundary, the aim is to determine the one whose inner points minimizes a functional. This minimization is performed using an iterative algorithm (e.g. L-BFGS or Truncated Newton) starting with an initial grid, usually constructed using an algebraic method, TFI for example ([9]). The Truncated Newton with Moré-Thuente's line search, has proven to be quite effective for this application ([3]).

2.2 The Classical Area Functional

According to the previous formulation, the area functional in its discrete version is

$$F_{A^2} = \sum_{q=1}^{N} \alpha_q^2$$

The following are the key properties of the area functional:Let Ω, m, n be fixed

1. *If there exists a grid G such that the triangles of every cell have the same area, then G is a critical point for the area functional.*

2. *The Hessian of the area functional valued in G is semipositive definite.*

The proof of these properties given in [2] is lengthly and makes extensive use of matrix algebra. A more general form of these will be stated later in this work; our proof is simpler and straightforward.

The properties just stated guarantee that if there exists an ideal grid for the area functional, then its minimization leads to it; if there is no such an ideal grid, the minimum will be reached in a grid that is as closest as possible to have triangles of constant area. The questions of the existence of an ideal grid for this functional depends strongly on the shape of the boundary and on the distribution of the points over it. In practice, we observe that optimal grids are far from being smooth and that convexity is attained just for simple regions.

2.3 The Length Functional

The simplest form of the discrete length functional is

$$F_l = \sum_{q=1}^{N} l(\Delta_q) = \sum_{q=1}^{N} l_q$$

this is to be used in order to keep our discussion on a basis as elementary as possible and because when it is used in combination with other functionals produces very good results. A more general form for the length functional is given in [2] and [1].

F_l is a quadratic functional, this makes its optimization quite efficient with almost any method. It also has the property that its hessian is positive definite everywhere ([2]), and this guarantees the existence and uniqueness of the minimum. It has been observed that:

1. *Optimal grids are quite smooth.*

2. *The number of iterations needed is very small compared with the corresponding to the area functional.*

3. *Optimal grids are frequently non convex.*

2.4 The Orthogonality Functional

Like its continuous analogue, the discrete version of the orthogonality functional used in an isolated manner does not produce good grids for almost any region; however its combinations with length and area functional do. The form for this functional is

$$\sum_{q=1}^{N} o(\Delta_q)^2 = \sum_{q=1}^{N} o_q^2$$

2.5 Combination of Functionals

In the discrete approach, it is also convenient to combine different functionals in order to calculate adequate grids. The general form of this combination is

$$F = \tau_1 F_{A^2} + \tau_2 F_l + (1 - \tau_1 - \tau_2) F_o$$

where $0 \leq \tau_1, \tau_2 \leq 1$. When these kind of combinations are formed, the functionals have to be properly scaled (see [14]).

2.5.1 The Area-Length Functionals

When we take $\tau_1 = \tau$, $\tau_2 = 1-\tau$ we get the family of functionals

$$F = \tau F_{A^2} + (1-\tau) F_l \qquad 0 \leq \tau \leq 1$$

that with convenient choices of parameter τ (usually τ close to 1) produces acceptable grids for most of the simple regions. It is a matter of experience to choose the best value of τ and to get good grids if we are dealing with a complicated region; this makes using the area-length functionals very complicated, even for experts in grid generation. The grids produced are smoother than the corresponding to the area functional, but they have more non convex cells. The number of iterations to reach convergence is considerably smaller than for the area funcional, even though they are greater than for the length functional.

2.5.2 The Area-Orthogonality Functional

A combination producing very good results for not too complicated regions is obtained with $\tau_1 = 1/2$; $\tau_2 = 0$. The resulting functional is called area-orthogonality and it is given by

$$F_{ao} = \frac{1}{2} F_{A^2} + \frac{1}{2} F_o$$

It is very surprising that being combination of two functionals not leading to smooth grids, the area-orthogonality functional produces relatively smooth meshes, the areas of the triangles have a small variation and the number of iterations is lesser than for the area functional; however in most of the regions convexity is not attained.

2.6 General Discrete Area Functionals

An expression of the form

$$F = \sum_{q=1}^{N} f(\alpha_q)$$

where, as before, N is the total number of triangles in the grid, α_q is twice the area of the triangle Δ_q, and f is of class C^2, is called *Discrete Area Functional*. Note that F is a function of the coordinates of the inner points of the grid; for the sake of notational simplicity, let us arrange these $2(m-2)(n-2)$ variables in a vector and call this vector z. In this section we state and prove the key properties of these functionals; however, first let us introduce the main ideas in an heuristic way.

Let us consider the following constrained minimization problem:

Minimize the function of N real variables

$$\sum_{q=1}^{N} f(w_q)$$

subject to constraint

$$\sum_{q=1}^{N} w_q = B$$

where B is constant and f is a function such that f' is 1-1, $f'' \geq 0$.

This constrained problem is equivalent to the following unconstrained:

Minimize

$$F(w_1,\ldots,w_{N-1}) = \sum_{q=1}^{N-1} f(w_q) + f\left(B - \sum_{q=1}^{N-1} w_q\right)$$

It's easy to see that

$$\frac{\partial F}{\partial w_i} = f'(w_i) - f'\left(B - \sum_{q=1}^{N-1} w_q\right) \quad i = 1,\ldots,N-1$$

The critical points $(\hat{w}_1, \hat{w}_2, \ldots, \hat{w}_{N-1})$ are such that

$$f'(\hat{w}_i) = f'\left(B - \sum_{q=1}^{N-1} \hat{w}_q\right) \qquad i = 1,2,\ldots,N-1$$

and thus f' being 1-1 implies

$$\hat{w}_i = B - \sum_{q=1}^{N-1} \hat{w}_q \qquad i = 1,2,\ldots,N-1$$

and from this

$$\hat{w}_1 = \hat{w}_2 = \ldots = \hat{w}_{N-1} = \frac{B}{N}$$

Now,

$$\frac{\partial^2 F}{\partial w_j \partial w_i} = \delta_{i,j} f''(w_i) + f''\left(B - \sum_{q=1}^{N-1} w_q\right)$$

and in the critical point this quantity is equal to

$$f''\left(\frac{B}{N}\right)[\delta_{i,j} + 1]$$

therefore the hessian in that point is

$$H(F) = f''\left(\frac{B}{N}\right)\begin{pmatrix} 2 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 1 & \ldots & 1 \\ 1 & 1 & 2 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 2 \end{pmatrix}$$

By hyphotesis $f''\left(\frac{B}{N}\right) \geq 0$, $H(F)$ will be semipositive definite if matrix

$$C = \begin{pmatrix} 2 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 1 & \ldots & 1 \\ 1 & 1 & 2 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 2 \end{pmatrix}$$

is positive (or semi-positive) definite. Now, it is easy to check that C is of full rank and has 1 and N as its only eigenvalues; therefore it is positive definite and function F has a minimum in the critical point.

If we think of w_q and B as α_q and $4Area(\Omega)$, respectively, the former result would tell us that almost any area functional would serve for the purpose of generating grids with almost uniform triangles areas.

Now, returning to our Area Functional F, let us observe that the r-th component of the gradient of F is

$$\frac{\partial F}{\partial z_r} = \sum_{q=1}^{N} f'(\alpha_q) \frac{\partial}{\partial z_r} \alpha_q$$

and the component r,s-th of the hessian of F is

$$\begin{aligned} \frac{\partial^2 F}{\partial z_s \partial z_r} &= \sum_{q=1}^{N} f''(\alpha_q) \left(\frac{\partial}{\partial z_s}\alpha_q\right)\left(\frac{\partial}{\partial z_r}\alpha_q\right) + \\ &\quad + f'(\alpha_q) \frac{\partial^2}{\partial z_s \partial z_r} \alpha_q \end{aligned}$$

Let us define

$$\begin{aligned} \Gamma &= diag(f''(\alpha_1), f''(\alpha_2), \ldots, f''(\alpha_N)) \\ A_{i,j} &= \frac{\partial}{\partial z_i}(\alpha_j) \\ B_q &= hessian\ of\ \alpha_q \end{aligned}$$

Then the hessian of F can be written as

$$H = A^t \Gamma A + \sum_{q=1}^{N} f'(\alpha_q) B_q$$

Next proposition states that, theoretically, almost every area functional is useful to generate grids with small area variations. As we have said, this is not true in practice.

Proposition 2.1. *If G is a grid such that*

$$\alpha_q = \bar{\alpha} \qquad q = 1, 2, \ldots, N$$

and F is an area functional, then

1. *G is a critical point for F.*
2. *If in addition f is such that $f'' \geq 0$, G is a minimum for F.*

Proof.

Because of

$$\sum_{q=1}^{N} \alpha_q = 4 \; Area(\Omega)$$

for every grid over Ω, we have that

$$\frac{\partial}{\partial z_r} \left(\sum_{q=1}^{N} \alpha_q \right) = 0$$

in particular

$$\frac{\partial}{\partial z_r} \left(\sum_{q=1}^{N} \alpha_q \right) (G) = 0 \quad r = 1, 2, \ldots, (m-2)(n-2)$$

For the same reason,

$$\frac{\partial^2}{\partial z_s \partial z_r} \left(\sum_{q=1}^{N} \alpha_q \right) (G) = 0$$

for $r, s = 1, \ldots, (m-2)(n-2)$

Thus

$$\begin{aligned} \left[\frac{\partial F}{\partial z_r} \right] (G) &= \left[\sum_{q=1}^{N} f'(\alpha_q) \frac{\partial}{\partial z_r} (\alpha_q) \right] (G) \\ &= \left[\sum_{q=1}^{N} f'(\bar{\alpha}) \frac{\partial \alpha_q}{\partial z_r} \right] (G) \\ &= f'(\bar{\alpha}) \left[\sum_{q=1}^{N} \frac{\partial \alpha_q}{\partial z_r} \right] (G) \\ &= 0 \end{aligned}$$

and the first assertion is true.

On the other hand, the s, r -th component of the second term of the expression for H evaluated in G is

$$f'(\bar{\alpha}) \left[\sum_{q=1}^{N} \frac{\partial^2 \alpha_q}{\partial z_s \partial z_r} \right] = 0$$

Because Γ is diagonal with non negative elements, the hessian of F in G is

$$\begin{aligned} H(G) &= \left[A^t \Gamma^{1/2} \Gamma^{1/2} A \right] (G) \\ &= \left[(\Gamma^{1/2} A)^t (\Gamma^{1/2} A) \right] (G) \\ &\geq 0 \end{aligned}$$

That is, H evaluated in G is semipositive definite. Therefore, G is a minimum for F.

3. SOME DISCRETE AREA FUNCTIONALS

In certain sense, these results have served as a guide to design new area functionals. We list here some examples and show grids generated using them, sometimes in combination with the length and orthogonality functionals in order to introduce some smoothness in optimal grids (Figs 1-12).

$$F_{A^4} = \sum_{q=1}^{N} (\alpha_q - \bar{\alpha})^4$$

This functional is intended to magnify deviations from the average area. Its minimization is a little more complicated than the corresponding to the classical functional, nevertheless results are better: optimal grids have a smaller area variation .

$$F_k = \sum_{q=1}^{N} \frac{1}{k + \alpha_q}$$

This is not a single functional, but a family of functionals depending in the parameter k. The parameter is chosen to make all the denominators positive; that is $k > -\alpha_{min}$. Once the minimization process is finished for a certain value of k, its value is update (diminished) in order to force the grid to be convex.

$$F_{k_1,k_2} = \sum_{q=1}^{n} \left[\frac{1}{k_1 + \alpha_q} + \frac{1}{k_2 - \alpha_q} \right]$$

Again the parameters k_1 and k_2 are such that all denominators are positive. Its minimization gives a grid such that its triangles have areas between k_1 and k_2 and sufficiently separated from them. An updating process on both parameters allows to construct convex grids with no large cells.

A detailed study of its theoretical properties can be found in ([14, 17]), for example. It is worth mentioning that the last two (families of) functionals produce convex grids. Another functional producing good results has been developed by Barrera and Domínguez ([7]).

4. SOME CONCLUSIONS

The results in sections 1.5 and 2.6 are of theoretical interest and have served as a guide to design more efficient grid-generation area functionals, both continuous and discrete. These state that a minimum is attained in grid with constant jacobian, in the continuous version, and in a grid with all the cells being of equal area, for the discrete ones. However for a given region is not known whether such a grid exists; and, even if the answer is in the positive sense, the numerical process of minimization could not be capable of reaching this ideal minimum.

The classical continuous area functional Φ_A has a natural discrete analog, namely F_{A^2}. In fact, all the classical discrete functionals share this property. With the design of new discrete area functionals such as $F_{A^4}, F_k, F_{k_1,k_2}$ the question arose whether corresponding continuous functionals could be constructed yielding good grids; unfortunately, this has not been the case: just translating the former into their continuous version has not produced the desired results. A deeper understanding is necessary. Regarding the exponential area functional introduced in ([7], the corresponding work is in process.

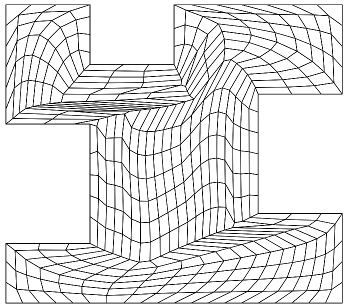

Figure 1. Region m19. F_{A^2}. Non-convex.

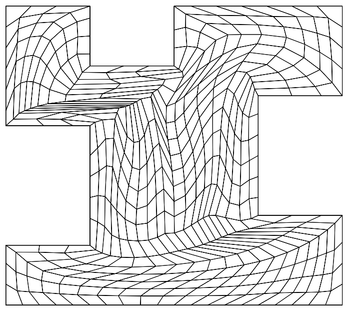

Figure 2. Region m19. F_{A^4}. Convex.

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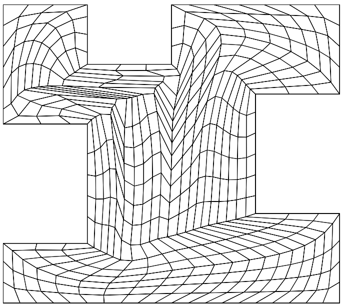

Figure 3. Region m19. F_k. Convex (but far from smooth).

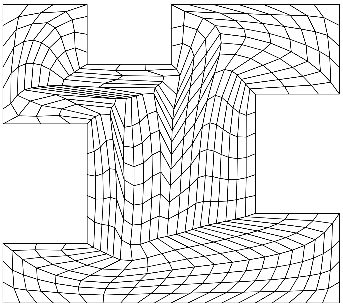

Figure 4. Region m19. F_{k_1,k_2}. Convex (and farther from smooth).

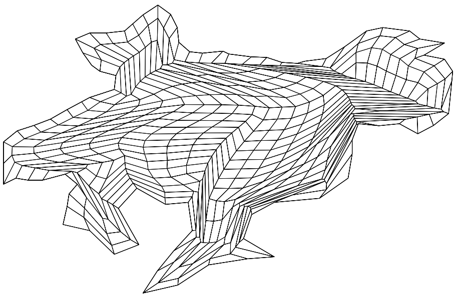

Figure 5. Region USSR. F_{A^2}. Far from convex.

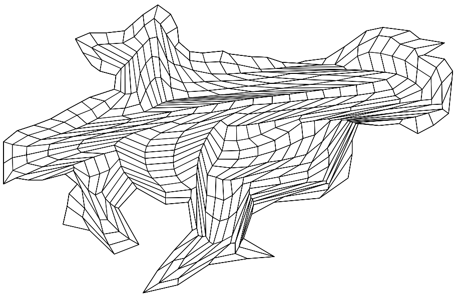

Figure 6. Region USSR. F_{A^4}. Non-convex.

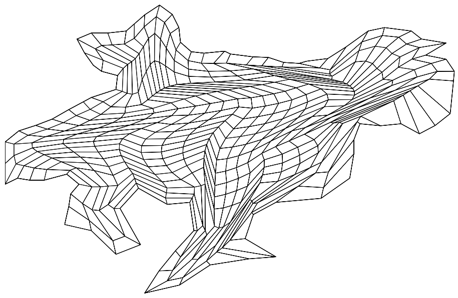

Figure 7. Region USSR. F_k. Convex.

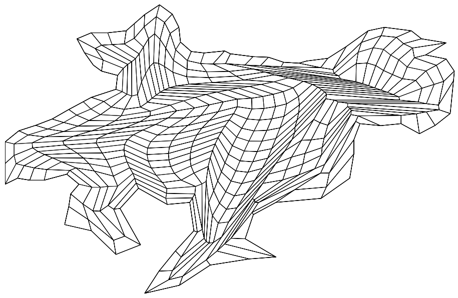

Figure 8. Region USSR. F_{k_1,k_2}. Convex.

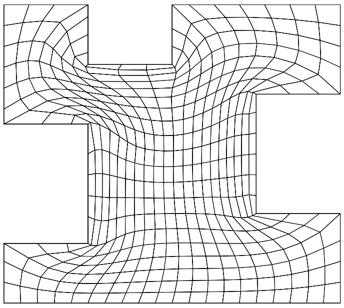

Figure 9. Region M19. F_{k_1,k_2} and length. Convex and Smooth.

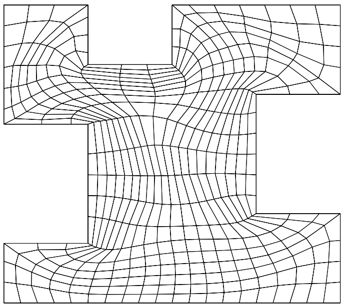

Figure 10. Region M19. F_{k_1,k_2} and orthogonality.Convex and Smooth.

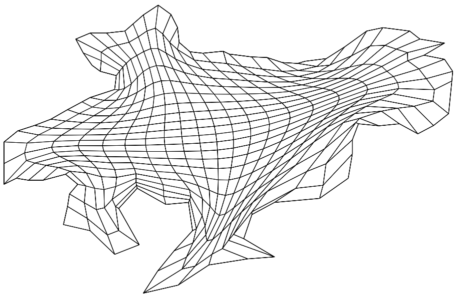

Figure 11. Region USSR. F_{k_1,k_2} **and length.Convex and Smooth.**

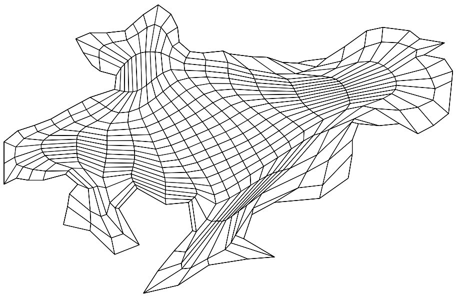

Figure 12. Region USSR. F_{k_1,k_2} **and orthogonality.Convex and Smooth.**

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