Parametrization of Plane Irregular Regions: A Semi-automatic Approach I

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Abstract In some problems, the solutions of partial differential equations use parametrizations of plane regions. However, it is difficult to get suitable parametrizations of irregular regions. In this paper we introduce a method for finding a parametrization of a polygonal region Ω. Our method decomposes Ω into a finite collection of admissible subregions. We use compatible parametrizations of these subregions to construct the parametrization of Ω as a block structured mesh.

1 Introduction

The parametrizations of irregular regions have many applications in Computer Aided Design, Engineering Modeling, and Shape Recognition [15]. The Finite Element Analysis and the Isogeometric Analysis use parametrizations of plane regions to solve partial differential equations [10].

Our UNAMALLA workgroup has generated bilinear and biquadratic B-splines parametrizations of simply connected regions using structured mesh generation [1, 2]. Nevertheless, some cells are elongated or skewed on highly irregular regions. This limitation can be overcome by splitting the region into subregions suitable for parametrization.

The problem we address is to decompose polygonal regions into admissible subregions and generate compatible parametrizations of these subregions. We want a method to solve this problem.

Researchers in Computer Aided Design have developed some methods for constructing compatible parametrizations of 2D regions. In that regard, Xu G. et al. [18, 19] have constructed high quality parametrizations by solving constrained optimization problems. Even so the overall process is computationally expensive for irregular regions.

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On the other hand, admissible subregions and their compatible parametrizations can be efficiently constructed by automatic methods of block structured mesh generation. Zint et al. [21] use triangulations for region decomposition, so the block connectivity depends on the triangulation. Bommes et al. [6] use the singularities of a cross-field to construct a family of parametrizations between coarse quad layouts of surfaces. Nevertheless, as the number of singularities increases, the number of layouts also increases. Recently, Xiao et al. [17] use the singular points of a cross-field to generate high quality structured meshes with smooth lines between the subregions. However, they provide examples in which the geometry is not irregular. On the other hand, Zhang et al. [20] decompose a polygonal region into a main block with multiple subregions organized in a hierarchy structure. Mesh generation is automatically carried out block by block from the main root block to the highest level. This method is not easy to extend to multiply-connected regions.

In this paper we propose a method to find both admissible subregions and their compatible parametrizations. Our method consists of two stages: region decomposition and parametrization construction.

Motivated by Liu et al.[13], we propose in Sect. 2 an interactive region decomposition method to obtain admissible subregions. In Sect. 3 the parametrization construction is carried out from the boundary to the interior of the subregions. First, the subregion boundaries are approximated by compatible B-splines curves in Sect. 3.1. Then, these curves are extended into the whole region by structured mesh generation in Sect. 3.2. Finally, we summarize the main steps of our method and show some examples to illustrate its robustness in Sect. 4.

2 Region Decomposition

Region decomposition is a fundamental step in our method. Let Ω be a a counter-clockwise oriented polygonal region. We decompose Ω into non-overlapping admissible polygons. A polygonal region is *admissible* if it has a parametrization such that its cells are rectangle-like quadrilaterals.

The key point of our region decomposition is the concavity. Lien [12] decomposes recursively a polygon into approximate convex polygons. He introduces concavity criteria to decompose polygons. Later, Liu et al. [13] proposed the *Dual-space Decomposition* (DUDE) to split polygons using their convex complements.

2.1 Concavity Measures and Admissible Regions

We use some concepts introduced in Lien [12] and Liu et al. [13] to measure the concavity of Ω. Let $H(\Omega)$ be the convex hull of Ω. The convex complement of Ω is $H(\Omega) \setminus \Omega$. The *bridges* of Ω are line segments contained in the convex complement of Ω which join two points in $\partial\Omega$. Each bridge β has a *pocket* ρ, that

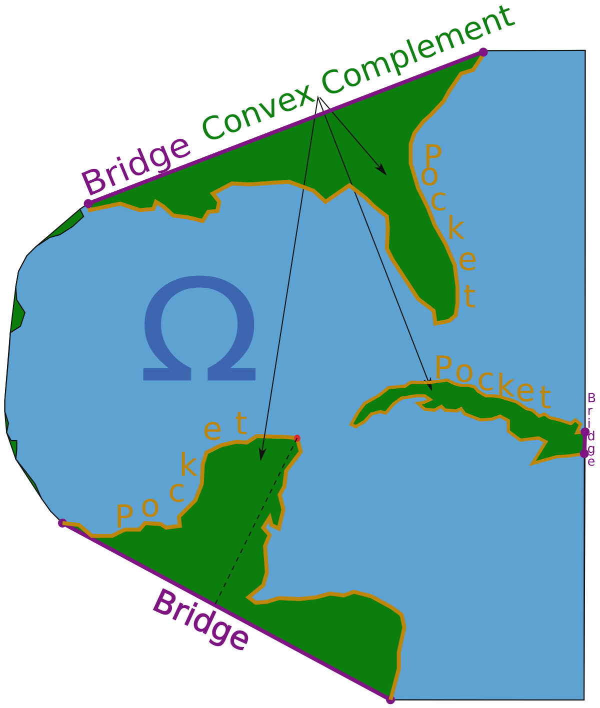

Fig. 1 Convex complement of the region `Gulf` and some of its bridges and pockets. The red point has the largest concavity in the corresponding pocket

is, the polygonal curve of $\partial\Omega$ with the smallest length which joins the ending points of β (Fig. 1).

We measure the concavity of pocket points and bridges. Let β be a bridge of Ω with pocket ρ. The concavity of a point $\boldsymbol{x}$ of ρ is the distance from $\boldsymbol{x}$ to β. This distance is the straight line distance to β or the arc length of the polygonal curve in ρ which joins $\boldsymbol{x}$ with the nearest ending point of β.

The concavity of a bridge is the largest concavity of its pockets points. We measure the concavity of the bridges using the straight line distance to the bridges. The largest concavity of the bridges of Ω is denoted by $c_1(\Omega)$. The region Ω is scaled inside a circle of radius one centered at the centroid of Ω so that $c_1(\Omega)$ be scale independent.

In addition to the concavity measures in [12, 13], we introduce the concavity measure

$$c_2(\Omega) := \frac{\text{Area}(H(\Omega)) - \text{Area}(\Omega)}{\text{Area}(H(\Omega))}. \tag{1}$$

This is the relative size of region with respect to its convex hull.

Regions with small concavity measures are suitable for parametrization, so we want subregions of Ω that satisfy the following concavity criteria:

- 1° *concavity criterion:* The concavity of the bridges of Ω is small.

$$c_1(\Omega) \leq \tau_1, \quad \tau_1 \in (0, 1). \tag{2}$$

- 2° *concavity criterion:* The area difference between the region and its convex hull is small.

$$c_2(\Omega) \leq \tau_2, \quad \tau_2 \in (0, 1). \tag{3}$$

Regions which satisfy both concavity criteria are admissible regions. So convex regions are admissible. On the other hand, non-convex regions which do not satisfy the previous criteria are decomposed into admissible subregions. We want an *admissible decomposition* of Ω, that is, a collection $\{\Omega_k\}_{k=1}^n$ of polygonal subregions of Ω such that

1. $\Omega = \cup_{k=1}^n \Omega_k$.
2. $\Omega_i \cap \Omega_j \neq \emptyset \Longrightarrow \Omega_i \cap \Omega_j \subset \partial\Omega_i \cap \partial\Omega_j \quad \forall i, j$.
3. Each Ω_k satisfies both concavity criteria (2)–(3).

2.2 Decomposition Method

Our decomposition method is interactive. It uses some ideas of DUDE [13]. The region Ω is recursively split into two subregions Ω_1 and Ω_2 by a cut when Ω does not satisfy the concavity criteria (2)–(3) (Fig. 2).

A *cut* of Ω is a line segment in the interior of Ω which joins two points of $\partial\Omega$. We make a cut in each step of our method. However, not just any cut separates Ω into admissible subregions. We propose an interactive cut choice method based on concavity measures. It consist of the following steps:

1. Compute the convex hull $H(\Omega)$
2. Find the pockets of Ω with ending points in the boundary of $H(\Omega)$. Select the pockets such that their union with the corresponding bridges have large area.
3. In each pocket compute the concavity of their points using either the straight line distance or the arc length. Select a point $\boldsymbol{x}_1$ with large concavity in a pocket ρ_1 as an ending point of the cut.
4. Select a point $\boldsymbol{x}_2$ with large concavity in a pocket different from ρ_1 such that the segment $\overline{\boldsymbol{x}_1\boldsymbol{x}_2}$ is a cut, else choose $\boldsymbol{x}_2$ as an intersection point of $\partial\Omega$ with the line P perpendicular to the bridge of ρ_1 which passes at $\boldsymbol{x}_1$. Otherwise choose $\boldsymbol{x}_2 \in \partial\Omega \setminus \rho_1$ with the smallest straight line distance to $\boldsymbol{x}_1$ such that the segment $\overline{\boldsymbol{x}_1\boldsymbol{x}_2}$ does not cross previous cuts of Ω (Fig. 3).

The segment $\overline{\boldsymbol{x}_1\boldsymbol{x}_2}$ is the chosen cut of Ω.

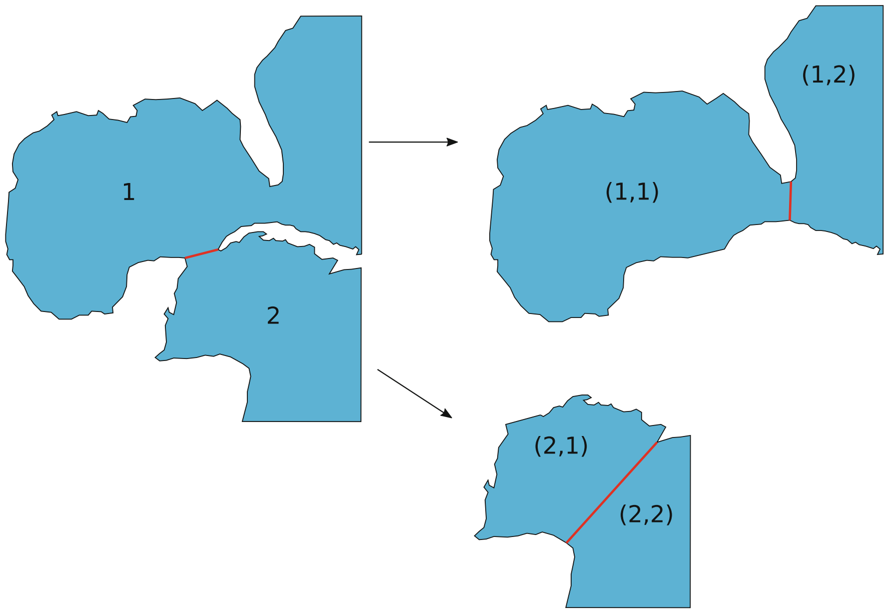

Fig. 2 The decomposition process of the region `Gulf`. The cuts are colored in each step and the subregions are labeled by 1–2 tuples to indicate a binary tree structure

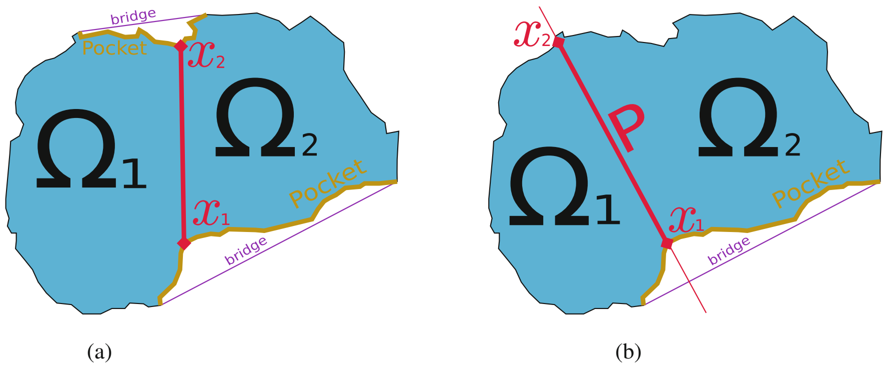

Fig. 3 Choices for a cut $\overline{\boldsymbol{x}_1\boldsymbol{x}_2}$ in our region decomposition method. (**a**) Pocket points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ with large concavity in different pockets. (**b**) A pocket point $\boldsymbol{x}_1$ with large concavity and the intersection $\boldsymbol{x}_2$ of $\partial\Omega$ and the line P

Our region decomposition method is shown in Algorithm 1. We interactively choose suitable cuts in each step. These cuts does not necessarily connect two pockets. On the other hand, the DUDE method joins points with large concavity measure and automatically selects the cuts using a triangulation.

Algorithm 1 Region decomposition method

```
procedure DECOMPOSITION(Ω)
    if Ω satisfies both concavity criteria (2)–(3) then
        return Ω
    else
        x1x2 ← Cut Choice Method(Ω)
        split Ω into Ω1 and Ω2 using the cut x1x2
        Decomposition(Ω1)
        Decomposition(Ω2)
    end if
end procedure
```

3 Parametrization Construction

Our next task is to find compatible parametrizations of admissible subregions. Any two parametrizations of different subregions are *compatible* if they have the same points on the intersection. First, we generate compatible parametrizations of the boundaries. Then, we extend these parametrizations into the interior of the subregions.

3.1 Compatible Parametrizations of the Boundaries

Let $\{\Omega_k\}_{k=1}^n$ be an admissible decomposition of Ω. We want to parametrize $\partial\Omega_k$ on the boundary of $R = [0, 1] \times [0, 1]$. To that end, we split $\partial\Omega_k$ into four consecutive polygonal curves $\Omega_{k,\text{bottom}}$, $\Omega_{k,\text{right}}$, $\Omega_{k,\text{top}}$ and $\Omega_{k,\text{left}}$ delimited by four points of $\partial\Omega_k$. The polygonal curves $\Omega_{k,\text{bottom}}$ and $\Omega_{k,\text{top}}$ are opposite boundaries of Ω_k. The same applies for $\Omega_{k,\text{right}}$ and $\Omega_{k,\text{left}}$.

Since the subregions Ω_k are admissible, we can manually choose four points of $\partial\Omega_k$ such that their interior angles are less than 180° and the opposite boundaries have approximately the same length. Zhang et al. [20] propose another choice for the four points.

We identify the decomposition cuts of Ω in the four boundary curves of each suregion, then we split each one of polygonal curves $\Omega_{k,\text{bottom}}$, $\Omega_{k,\text{right}}$, $\Omega_{k,\text{top}}$ and $\Omega_{k,\text{left}}$ into consecutive polygonal sections which are either cuts of Ω or maximal polygonal curves contained in $\partial\Omega$. We enumerate these polygonal sections starting with $\Omega_{k,\text{bottom}}$, then the sections in $\Omega_{k,\text{right}}$ and $\Omega_{k,\text{top}}$, and finally those in $\Omega_{k,\text{left}}$ for $k = 1, \dots, n$. These sections form a polygonal decomposition $\{Q_p\}_{p=1}^s$ of $\cup_{k=1}^n \partial\Omega_k$ (Fig. 4).

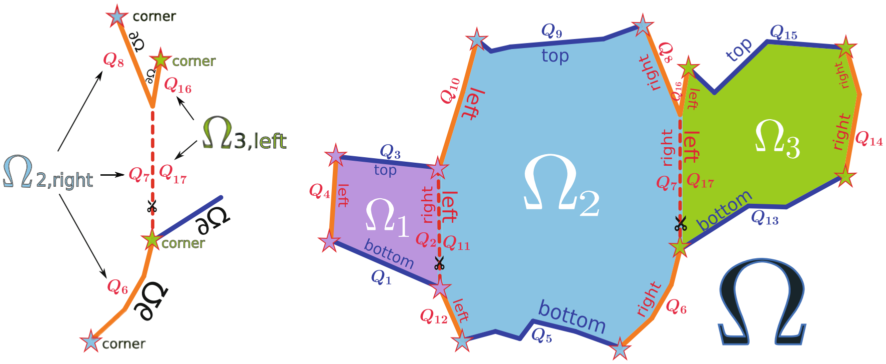

Fig. 4 The region Ω is split into Ω_1, Ω_2 and Ω_3. We split $\partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3$ into 17 polygonal sections Q_p. The sections Q_2, Q_7, Q_{11} and Q_{17} are cuts of Ω while the other sections are contained in $\partial\Omega$. The boundary curve $\Omega_{2,\text{right}}$ is split into three sections while $\Omega_{3,\text{left}}$ is split into two sections

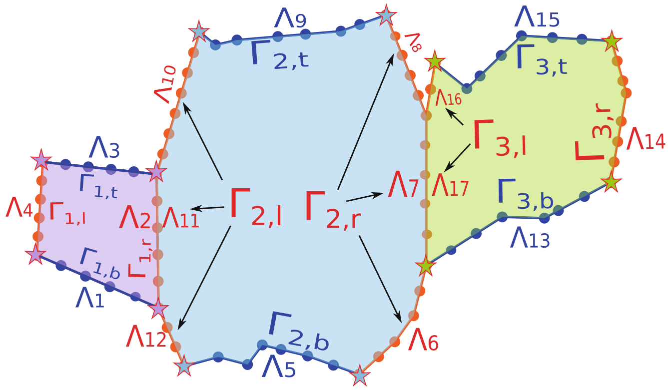

Fig. 5 The sections of the boundaries in Fig. 4 are reparametrized as polygonal curves Λ_p and the four boundary curves of each subregion are reparametrized as $\Gamma_{k,b}$, $\Gamma_{k,r}$, $\Gamma_{k,t}$ and $\Gamma_{k,l}$

Each polygonal section Q_p is reparametrized with respect to arc-length as a polygonal curve Λ_p with equidistant points. We join the sections Λ_p to obtain reparametrizations of $\Omega_{k,\text{bottom}}$, $\Omega_{k,\text{right}}$, $\Omega_{k,\text{top}}$ and $\Omega_{k,\text{left}}$ denoted by $\Gamma_{k,\text{b}}$, $\Gamma_{k,\text{r}}$, $\Gamma_{k,\text{t}}$ and $\Gamma_{k,\text{l}}$, respectively (Fig. 5).

Afterwards, we generate four uniform linear B-spline curves:

$$\begin{aligned}
\psi_{k,\text{bottom}} &\colon [0,1] \to \Gamma_{k,\text{b}},\\
\psi_{k,\text{right}} &\colon [0,1] \to \Gamma_{k,\text{r}},\\
\psi_{k,\text{top}} &\colon [0,1] \to \Gamma_{k,\text{t}},\\
\psi_{k,\text{left}} &\colon [0,1] \to \Gamma_{k,\text{l}}.
\end{aligned}$$

The control points of these B-spline curves are given by their polygonal curves. We combine these curves to get parametrizations $\psi_k : \partial R \to \partial\Omega_k$.

We want to extend ψ_k to the interior of Ω_k as explained by the UNAMALLA workgroup [1]. To that end, $\psi_{k,\text{bottom}}$ and $\psi_{k,\text{top}}$ must have the same number of points and the same condition on $\psi_{k,\text{right}}$ and $\psi_{k,\text{left}}$, that is,

$$\text{number of points of } \Gamma_{k,\text{b}} = \text{number of points of } \Gamma_{k,\text{t}}, \tag{4}$$

$$\text{number of points of } \Gamma_{k,\text{r}} = \text{number of points of } \Gamma_{k,\text{l}}. \tag{5}$$

Let m_p be the number of points of Λ_p. In order to formulate Eqs. (4) and (5) in terms of m_p we identify the sets of indexes of the polygonal sections Λ_p in each subregion by introducing some notation.

Let $s_{k,b}$, $s_{k,r}$, $s_{k,t}$ and $s_{k,l}$ be the number of polygonal sections Λ_p in $\Gamma_{k,\text{b}}$, $\Gamma_{k,\text{r}}$, $\Gamma_{k,\text{t}}$ and $\Gamma_{k,\text{l}}$, respectively. Denote by s_k the number of polygonal sections Λ_p in $\Gamma_{k,\text{b}} \cup \Gamma_{k,\text{r}} \cup \Gamma_{k,\text{t}} \cup \Gamma_{k,\text{l}}$. Let

$$\sigma_k = \begin{cases} 0, & \text{if } k = 1; \\ \sum_{\ell=1}^{k-1} s_\ell, & \text{if } k > 1; \end{cases} \qquad k = 1, \ldots, n.$$

We define the set of indexes

$$J_{k,b} = \sigma_k + \left\{1, \ldots, s_{k,b}\right\},$$

where the sum means that σ_k is added to each element of the other set. Similarly, we define

$$J_{k,r} = \sigma_k + \left\{s_{k,b} + 1, \ldots, s_{k,b} + s_{k,r}\right\},$$
$$J_{k,t} = \sigma_k + \left\{s_{k,b} + s_{k,r} + 1, \ldots, s_{k,b} + s_{k,r} + s_{k,t}\right\},$$
$$J_{k,l} = \sigma_k + \left\{s_{k,b} + s_{k,r} + s_{k,t} + 1, \ldots, s_k\right\}.$$

Then, Eqs. (4)–(5) are formulated as

$$\sum_{j \in J_{k,b}} m_j - s_{k,b} = \sum_{j \in J_{k,t}} m_j - s_{k,t}, \tag{6}$$

$$\sum_{j \in J_{k,r}} m_j - s_{k,r} = \sum_{j \in J_{k,l}} m_j - s_{k,l}. \tag{7}$$

We want compatible parametrizations of the boundaries. So they must have the same points in the intersections. By construction, the intersections of the subregions are cuts of Ω. Since we have n subregions, there are $n - 1$ cuts. Let $\{c_i\}_{i=1}^{n-1}$ be the set of the decomposition cuts of Ω. For each cut c_i we have exactly two polygonal sections of $\cup_{k=1}^{n} \partial\Omega_k$ which coincide with c_i.

Let us remember that s is the number of polygonal sections Λ_p in $\cup_{k=1}^{n}\partial\Omega_k$. Let $\gamma_i, \delta_i \in \{1, \ldots, s\}$ be the indexes of two sections which coincide with c_i. Then $\psi_1, \ldots, \psi_n$ are compatible if

$$m_{\gamma_i} = m_{\delta_i}, \quad i = 1, \ldots, n-1. \tag{8}$$

Let $q = 3n - 1$. We put together Eqs. (6)–(8) as the system of linear equations

$$A\boldsymbol{m} = \boldsymbol{b}, \tag{9}$$

where $A \in \mathbb{Z}^{q\times s}$ with entries given by

$$a_{2k-1,j} = \begin{cases} 1, & \text{if } j \in J_{k,b}; \\ -1, & \text{if } j \in J_{k,t}; \\ 0, & \text{otherwise}; \end{cases} \qquad \begin{array}{l} k = 1, \ldots, n, \\ j = 1, \ldots, s. \end{array}$$

$$a_{2k,j} = \begin{cases} 1, & \text{if } j \in J_{k,r}; \\ -1, & \text{if } j \in J_{k,l}; \\ 0, & \text{otherwise} \end{cases} \qquad \begin{array}{l} k = 1, \ldots, n, \\ j = 1, \ldots, s. \end{array}$$

$$a_{2n+i,j} = \begin{cases} 1, & \text{if } j = \gamma_i; \\ -1, & \text{if } j = \delta_i; \\ 0, & \text{otherwise} \end{cases} \qquad \begin{array}{l} i = 1, \ldots, n-1, \\ j = 1, \ldots, s, \end{array}$$

$\boldsymbol{b}$ is a vector in $\mathbb{Z}^q$ with entries given by

$$\begin{aligned} b_{2j-1} &= s_{j,b} - s_{j,t}, && j = 1, \ldots, n; \\ b_{2j} &= s_{j,r} - s_{j,l}, && j = 1, \ldots, n; \\ b_j &= 0, && j = 2n+1, \ldots, 3n-1, \end{aligned}$$

and

$$\boldsymbol{m} = \left[m_1 \cdots m_s\right]^T.$$

The system of linear equations (9) is underdetermined since the matrix A has $3n - 1$ rows and for each subregion there are at least four boundary curves, i.e., there are at least $4n$ variables.

The vector $\boldsymbol{m}$ can be chosen as the solution of a linear integer programming problem:

$$\min\left\{ \mathbf{1}^T\mathbf{m} : \mathbf{m} \in \mathbb{Z}^s, A\mathbf{m} = \boldsymbol{b} \right\}, \tag{10}$$

where $\mathbf{1}$ is the vector of s ones. However, some entries of the optimal solution of the problem (10) could be negative or zero.

Let ℓ_p be the length of the polygonal section Λ_p. We choose compatible number of points so that their sum is minimized and the boundary point distribution depends on ℓ_p. Let $\ell_{\min}$ be the length of the smallest Λ_p. Given $K \in \mathbb{N}$, we measure the proportion of ℓ_p in comparison to $\ell_{\min}$ by

$$L_p = K \left\lceil \frac{\ell_p}{\ell_{\min}} \right\rceil. \tag{11}$$

Since the number of boundary points in Λ_p is m_p, we distribute at least L_p points in Λ_p by adding the constraint $m_p > L_p$ to the problem (10). So we solve the following integer linear programming problem:

$$\min \left\{ \mathbf{1}^T \mathbf{m} : \mathbf{m} \in \mathbb{Z}^s, A\mathbf{m} = \boldsymbol{b}, m_p \geq L_p \; p = 1, \dots, s \right\}. \tag{12}$$

By construction, $\boldsymbol{b}$ is an integer vector and A is a totally unimodular matrix, that its, all its square submatrices have determinant 0, 1 or -1. Therefore, the integer programming problem (12) is feasible by the Hoffmann-Kruskal theorem [14].

We get compatible mesh sizes by solving the problem (12). Other authors use a tree structure of the region decomposition [20]. The parametrizations $\psi_{k,\text{bottom}}$, $\psi_{k,\text{right}}$, $\psi_{k,\text{top}}$ and $\psi_{k,\text{left}}$ with these mesh sizes are compatible.

3.2 Parametrizations of the Subregions

Now, we extend the parametrization ψ_k into the interior of Ω_k. Let M_k be the number of points of $\Gamma_{k,\text{b}}$, and let N_k be the number of points of $\Gamma_{k,\text{r}}$ We generate convex structured quadrilateral meshes

$$G_k = \left\{ P_{i,j}^{(k)} \in \overline{\Omega_k} : i = 1, \dots, M_k, \; j = 1, \dots, N_k \right\}$$

such that

$$\text{points of } \Gamma_{k,\text{b}} = \left\{ P_{i,1}^{(k)} : i = 1, \dots, M_K \right\},$$

$$\text{points of } \Gamma_{k,\text{r}} = \left\{ P_{M_k,j}^{(k)} : j = 1, \dots, N_K \right\},$$

$$\text{points of } \Gamma_{k,\text{t}} = \left\{ P_{i,N_k}^{(k)} : i = 1, \dots, M_K \right\},$$

$$\text{points of } \Gamma_{k,\text{l}} = \left\{ P_{1,j}^{(k)} : j = 1, \dots, N_K \right\}.$$

The meshes G_k are automatically generated using the discrete variational approach of our UNAMALLA workgroup [3, 4, 16]. Garanzha [8] and Ivanenko [11] generate quadrilateral meshes with boundary adaptability.

By construction, the boundary points of G_k are the control points of the linear B-spline curves $\psi_{k,\text{bottom}}$, $\psi_{k,\text{right}}$, $\psi_{k,\text{top}}$ and $\psi_{k,\text{left}}$. Since the polygonal curves $\Gamma_{k,\text{b}}$, $\Gamma_{k,\text{r}}$, $\Gamma_{k,\text{t}}$ and $\Gamma_{k,\text{l}}$ satisfy Eqs. (4) and (5), the meshes $G_1, \ldots, G_n$ are compatible and their union is a block structured mesh G on Ω.

We use G_k to extend the boundary parametrization into the interior of Ω_k following the approach of the UNAMALLA workgroup [1]. Let B^2_{i,M_k} be the i-th linear B-spline with knot sequence given by a uniform partition of [0, 1] with M_k elements for $i = 1, \ldots, M_k$, and let B^2_{j,N_k} be the j-th linear B-spline with knot sequence given by a uniform partition of [0, 1] with N_k elements for $j = 1, \ldots, N_k$. We use the parametrizations $\varphi_k : R \to \Omega_k$ given by the bilinear tensor product B-spline

$$\varphi_k(\xi, \eta) = \sum_{i=1}^{M_k} \sum_{j=1}^{N_k} P^{(k)}_{i,j} B^2_{i,M_k}(\xi) B^2_{j,N_k}(\eta), \qquad \xi, \eta \in [0, 1]. \tag{13}$$

Let us make a few observations of these parametrizations:

- The control points of φ_k are the points of G_k.
- The map φ_k is 1-1 since all the cells of G_k are convex [1].
- By construction,

$$\varphi_k|_{\partial\Omega_k} \equiv \psi_k. \tag{14}$$

Moreover, since $\psi_1, \ldots, \psi_n$ are compatible, then $\varphi_1, \ldots, \varphi_n$ are compatible.

Therefore, we have decomposed the region Ω into a set of admissible regions $\Omega_1, \ldots, \Omega_n$, and we have constructed a family of compatible and admissible parametrizations $\varphi_1, \ldots, \varphi_n$ for these subregions given by (13).

4 Summary and Examples

We summarize the key points of our methodology:

1. Get an admissible decomposition $\{\Omega_k\}_{k=1}^n$ of Ω by Algorithm 1.
2. In each Ω_k choose four points as explained in Sect. 3.1.
3. Identify the cuts of Ω in each $\partial\Omega_k$ and split $\cup_{k=1}^n \partial\Omega_k$ into polygonal sections. Reparametrize these sections and join them to obtain reparametrizations $\Gamma_{k,\text{b}}$, $\Gamma_{k,\text{r}}$, $\Gamma_{k,\text{t}}$ and $\Gamma_{k,\text{l}}$ of the four boundary curves.
4. Solve the integer linear programming problem (12) to get compatible number of points for $\Gamma_{k,\text{b}}$, $\Gamma_{k,\text{r}}$, $\Gamma_{k,\text{t}}$ and $\Gamma_{k,\text{l}}$.
5. Generate convex structured quadrilateral meshes G_k on Ω_k such that their boundary points are the points of $\Gamma_{k,\text{b}}$, $\Gamma_{k,\text{r}}$, $\Gamma_{k,\text{t}}$ and $\Gamma_{k,\text{l}}$.
6. Construct parametrizations φ_k using the bilinear tensor product B-spline (13).

We generate parametrizations of four irregular polygonal regions using our method. First, the region decomposition is interactively carried out using our subroutines in JULIA [5] with concavity criteria tolerances $\tau_1 = 0.1$ and $\tau_2 = 0.45$. We do not use the DUDE code. Then, a Julia interface of the COIN-OR Branch and Cut solver [7] is used to solve the integer programming problem (12). We choose $K = 2$ for the lower bound L_p given by (11). Finally, automatic mesh generation is carried out by our UNAMALLA software [16] using a convex combination of the weighted discrete functionals H_ω and Area-Orthogonality so that mesh cells are accumulated in the boundary of the subregions [4, 9].

The region decomposition of the four polygonal regions and their corresponding block structured meshes are shown in Figs. 6, 7, 8, and 9. Table 1 shows the number of points, subregions and polygonal sections Λ_p of each region.

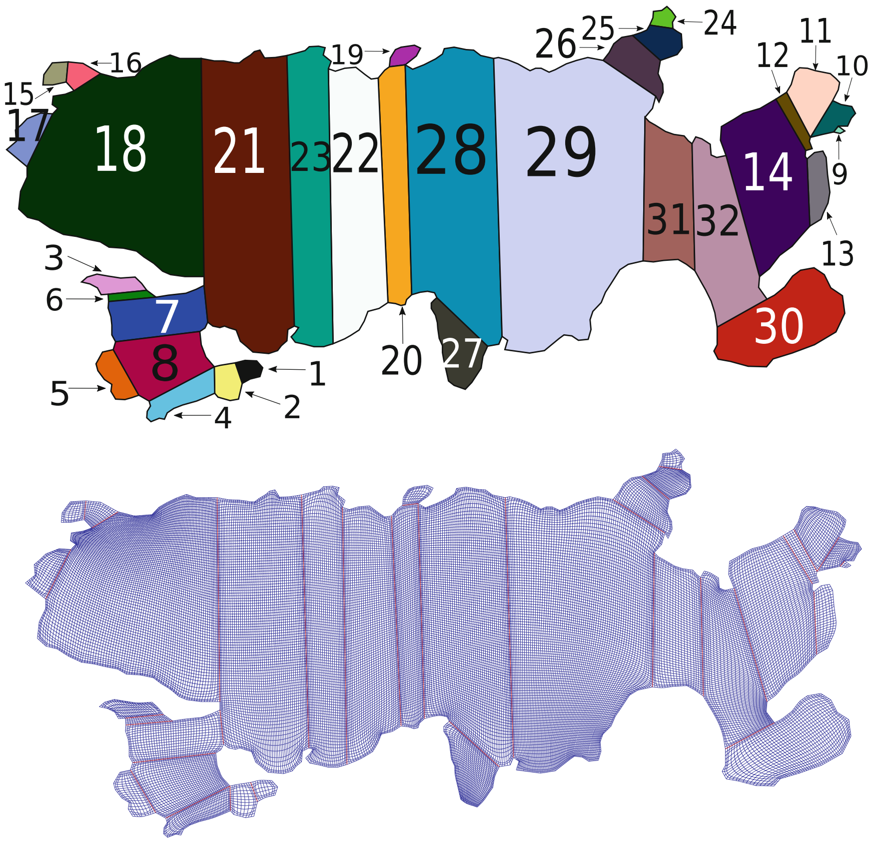

Fig. 6 Admissible decomposition of `Titicaca` and its block structured mesh

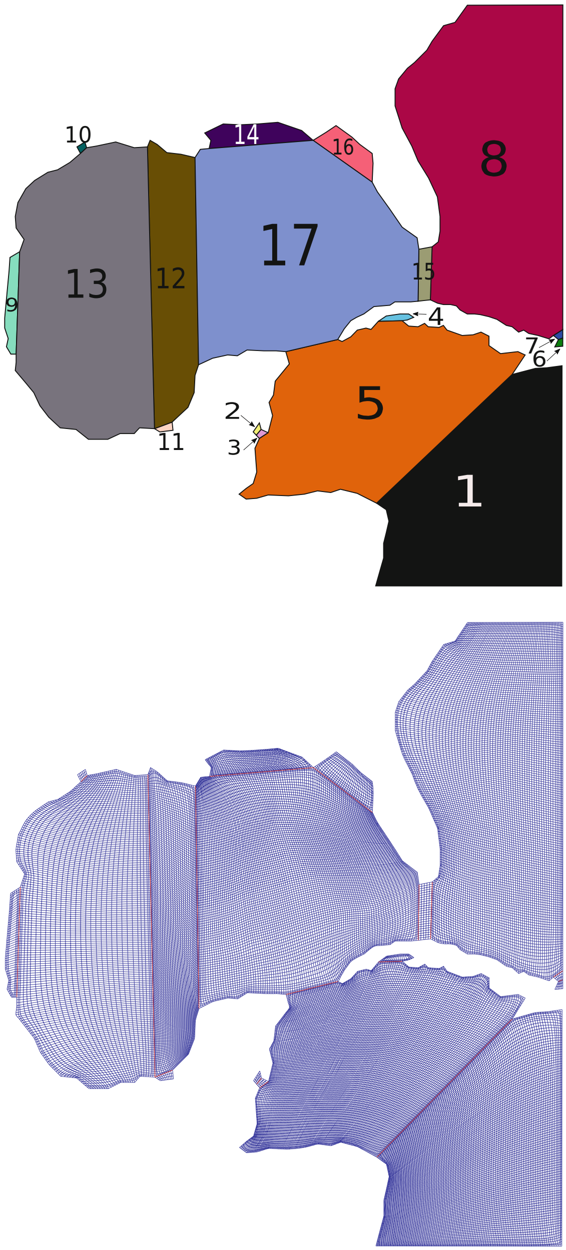

Fig. 7 Admissible decomposition of the region `Gulf` and its block structured mesh

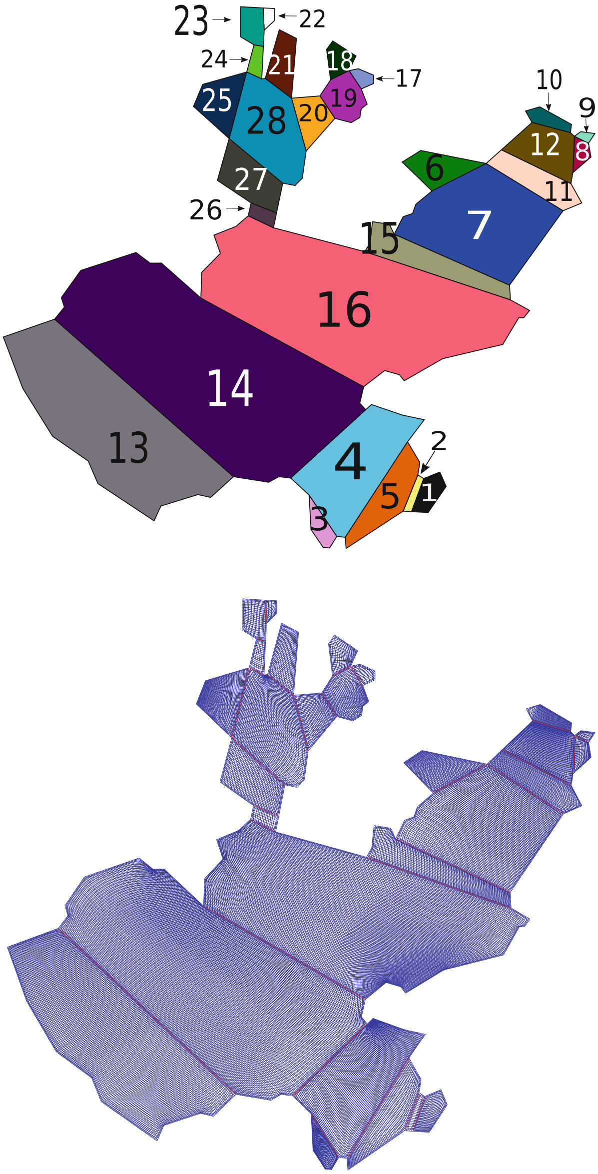

Fig. 8 Admissible decomposition of the region `Jalisco` and its block structured mesh

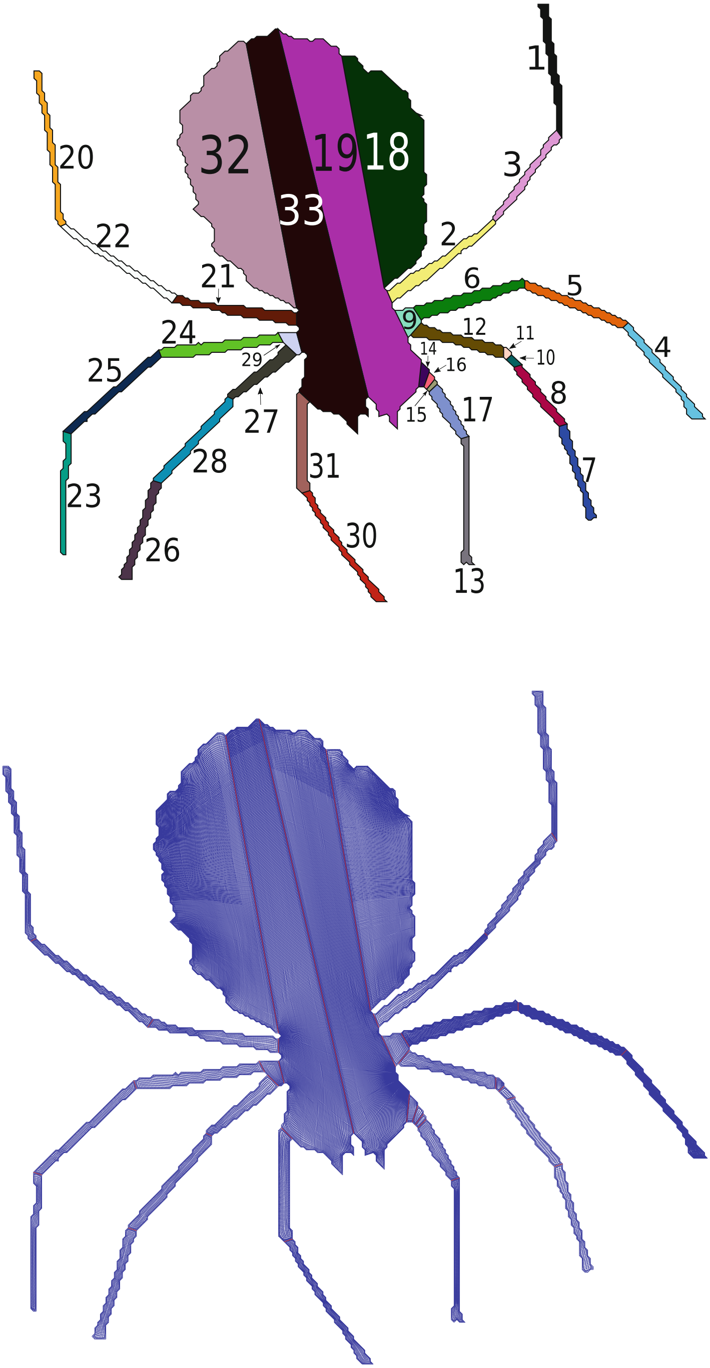

Fig. 9 Admissible decomposition of the region `Spider` [13] and its block structured mesh

Table 1 Number of points, subregions and polygonal sections in our example regions

Region	Number of points	Number of subregions	Number of polygonal sections Λ_p
`Gulf` Gulf of Mexico	199	17	88
`Titicaca` Lake Titicaca in Peru	365	32	161
`Jalisco` State of Jalisco in Mexico	120	28	139
`Spider` [13]	1388	33	150

5 Conclusions and Future Work

We have proposed a methodology to find a decomposition of an irregular polygonal region into admissible subregions and a family of compatible bilinear B-spline parametrizations of these subregions.

Irregular regions are interactively decomposed into admissible subregions using concavity measures. Then, the subregion boundaries are parametrized by compatible linear B-spline curves. Afterwards, these parametrizations are extended into the interior of the subregions as compatible bilinear B-splines by automatic structured mesh generation.

We plan to measure the quality of our meshes using the quality measures reported by UNAMALLA [9]. Our parametrizations are compatible, but they are not necessarily smooth between the subregions. We address this issue later.

We want to extend our method to multiply-connected plane regions. Our results would be submitted in the part II of this paper. We thank the anonymous referees that help us to make significant changes to improve our paper.

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