# Facultad de Ciencias

# "Caos en Ecuaciones Diferenciales Parciales (Solitones Caóticos)"

Rafael Pérez Pascual, Jorge Fujioka, Emilio Cortés, Rosalío Rodríguez, Aurea Espinosa, B.A. Malomed.

IF-UNAM, UAM-I, Tel Aviv University

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UNAM

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".

John Scott Russell

# REPORT

OF THE

# FOURTEENTH MEETING

OF THE

# BRITISH ASSOCIATION

FOR THE

# ADVANCEMENT OF SCIENCE;

HELD AT YORK IN SEPTEMBER 1844.

LONDON:

JOHN MURRAY, ALBEMARLE STREET.

1845.



## *Report on Waves.* By J. SCOTT RUSSELL, *Esq., M.A., F.R.S. Edin.*, *made to the Meetings in* 1842 *and* 1843.

*Members of Committee* { Sir JOHN ROBISON*, *Sec. R.S. Edin.* ; J. SCOTT RUSSELL, *F.R.S. Edin.* }

A PROVISIONAL Report on this subject was presented to the Meeting held at Liverpool in 1838, and is printed in the Sixth Volume of the Transactions. That report was a partial one. It states that "the extent and multifarious nature of the subjects of inquiry have rendered it impossible to terminate the examination of all of them in so short a time; but it is their duty to report the progress which they have made, and the partial results they have already obtained, leaving to the reports of future years such portions of the inquiries as they have not yet undertaken."

The first of these subjects of inquiry is stated to have been "to determine the varieties, phænomena and laws of waves, and the conditions which affect their genesis and propagation."

It is this branch of the duty of the Committee which forms the subject of the present report. Ever since the date of that report, it has happened that the author of this has been so fully pre-occupied by inevitable duty, that it was not in his power to indulge much in the pleasures of scientific inquiry; and as the active part of the investigation necessarily devolved upon him, it was not practicable to continue the series of researches on the ample and systematic scale originally designed, so soon as he had anticipated, so that the former report has necessarily been left in a fragmentary state till now.

But I have never ceased to avail myself of such opportunities as I could contrive to apply to the furtherance of this interesting investigation. I have now fully discussed the experiments which the former report only registered. I have repeated the former experiments where their value seemed doubtful, I have supplemented them in those places where examples were wanting. I have extended them to higher ranges, and where necessary to a much larger scale. In so far as the experiments have been repeated and more fully discussed, they have tended to confirm the conclusions given in the former report, as well as to extend their application.

The results here alluded to are those which concern especially the velocity and characteristic properties of the solitary wave, that class of wave which the writer has called the great wave of translation, and which he regards as the primary wave of the first order. The former experiments related chiefly to the mode of genesis, and velocity of propagation of this wave. They led to this expression for the velocity in all circumstances,

$$v = \sqrt{g(h+k)},$$

$k$ being the height of the crest of the wave above the plane of repose of the fluid, $h$ the depth throughout the fluid in repose, and $g$ the measure of gravity. Later discussions of the experiments not only confirm this result, but are themselves established by such further experiments as have been recently instituted, so that this formerly obtained velocity may now be regarded as the phænomenon characteristic of the wave of the first order.

The former series of experiments also contained several points of research not published in the former report, because not sufficiently extended to be of

* I cannot allow these pages to leave my hands without expressing my deep regret that the death of Sir John Robison has suddenly deprived the Association of a zealous and distinguished office-bearer, and myself of a kind friend. In all these researches the responsible duties were mine, and I alone am accountable for them; but in forwarding the objects of the investigation I always found him a valuable counsellor and a respected and cordial cooperator.

1995 junto al canal original “Edinburgh and Glasgow Union Canal” de 50 Km de largo. 89.3m de largo por 4.13m de ancho, 1.52m de profundidad.

Para un fluido ideal las ecuaciones para el campo de velocidades son

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0.$$

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \mathrm{grad})\, \mathbf{v} \right) = -\mathrm{grad} p + \mathbf{f}.$$

$$\frac{\partial \rho s}{\partial t} + \mathrm{div}(\rho s \mathbf{v}) = 0.$$

Si el flujo es incompresible, la fuerza externa es la gravedad y lo tomamos irrotacional

$$\mathbf{v} = \mathrm{grad}\phi.$$

$$\mathrm{div}(\mathrm{grad}\phi) = \nabla\phi = 0.$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] + \frac{p}{\rho} + gz = C.$$

Condiciones a la frontera.

$$\text{sobre la superficie} \quad z = \zeta(x,t) \qquad p = p_0.$$

$$v_z = \frac{\partial \phi}{\partial z} = 0 \quad \text{si} \quad z = -h.$$

Si $h \longrightarrow \infty \qquad \partial\phi/\partial x = \partial\phi/\partial z = 0.$

Aproximación lineal, despreciamos términos de segundo orden,

$$\zeta(x, t=0) \sim \cos kx.$$

$$\phi = A\exp(kz)\text{sen}\left(kx - \sqrt{gk}\,t\right)$$

$$\zeta = A\sqrt{\frac{k}{g}}\cos\left(kx - \omega t\right)$$

$$\omega^2 = gk, \qquad v_f = \sqrt{g/k} = \sqrt{g\lambda/2\pi}, \qquad v_g = \frac{1}{2}\sqrt{g/k} = \frac{1}{2}v_f.$$

Esta aproximación lineal es valida si

$$\frac{1}{2}Ak\sqrt{\frac{k}{g}} \ll 1.$$

En el caso de profundidad finita $h$, pero conservando la aproximación lineal,

$$\zeta = A\sqrt{\frac{k}{g}}\cos\left(kx - \omega t\right)$$

$$\phi = \sqrt{\frac{g}{k}} A \cosh k(z+h) \mathrm{sen}(kx - \omega t), \qquad z > -h$$

$$\omega^2 = gk\tanh(kh).$$

$$v_f = \left[\frac{g}{k}\tanh(kh)\right]^{1/2} = \sqrt{gh}\left(1 - \frac{1}{6}(kh)^2 + \cdots\right)$$

Efecto de los términos no lineales. Para ello aproximamos **v** a tercer orden en la profundidad $y = h + z$.

$$v_x = \frac{\partial \phi}{\partial x} = f(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3 + O(y^4)$$

$$v_z = \frac{\partial \phi}{\partial z} = g_1(x)y + g_2(x)y^2 + g_3(x)y^3 + O(y^4)$$

con la condición de diferenciabilidad. Substituimos en la ecuación de Laplace y Obtenemos a orden tres que

$$v_x = \frac{\partial \phi}{\partial x} = f - \frac{1}{2}\frac{\partial^2 f}{\partial x^2}y^2$$

$$v_z = \frac{\partial \phi}{\partial z} = \frac{\partial f}{\partial x}y - \frac{1}{6}\frac{\partial^3 f}{\partial x^3}y^3$$

Estas “soluciones” de la ecuación de Laplace las usamos para tomar en consideración el término no lineal de la de Euler y las condiciones a la frontera y las resolvemos a tercer orden.

Finalmente

$$\alpha\frac{\partial u}{\partial \tau}+\frac{3}{2}\alpha u\frac{\partial u}{\partial \xi}+\frac{1}{6}\beta^2\frac{\partial^3 u}{\partial \xi^3}=0$$

donde $u$ es la altura de la superficie normalizada por la amplitud $a$ de la onda, $\alpha$ y $\beta$ son parámetros libres que deben ser pequeños y se definen por

$$\alpha=\frac{a}{h} \qquad \beta=\frac{h}{\lambda}$$

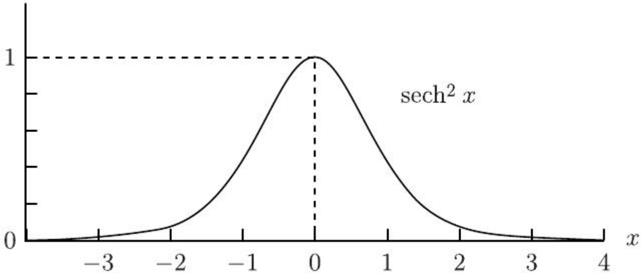
con $\lambda$ una anchura típica de la onda y

$$\xi=\sqrt{\frac{\alpha}{\beta}}\frac{x-ct}{\lambda} \qquad \tau=\sqrt{\frac{\alpha}{\beta}^3}\frac{ct}{\lambda}$$

$$\frac{\partial w}{\partial \tau}+w\frac{\partial w}{\partial \xi}+\frac{1}{6}\frac{\partial^3 w}{\partial \xi^3}=0 \quad \text{KdV}$$

una solución

$$w=A\text{sech}^2\left[\sqrt{\frac{A}{2}}\left(\xi-\frac{A}{3}\tau\right)\right]$$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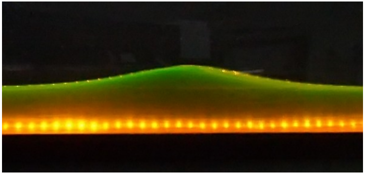


Zakharov 1968

$$\zeta = A\sqrt{\frac{k}{g}}\cos\left(kx - \omega t\right)$$

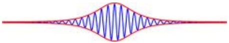
$$\zeta = A(x,t)\cos\left(kx - \omega t + \theta(x,t)\right)$$

Coordenadas transladandose con la velocidad de grupo

$$X = kx - \frac{1}{2}\omega t \qquad T = \frac{1}{4}\omega t$$

$$\psi = A\exp(i\theta)$$

$$i\frac{\partial\psi}{\partial T} + \frac{1}{2}\frac{\partial^2\psi}{\partial X^2} + 2k^2\left|\psi\right|^2\psi = 0$$



Pulsos de luz que se propagan por una fibra óptica. (medio no lineal Kerr)

$$E(z,t) = \frac{1}{2}u(z,T)\exp i(kz - \omega T) + c.c.$$

$$i\left[\frac{\partial u}{\partial z} + \frac{1}{v_g}\frac{\partial u}{\partial T}\right] + \epsilon\frac{\partial^2 u}{\partial T^2} + \gamma\left|u\right|^2 u = 0$$

$$t = T - \frac{1}{v_g}z$$

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial z^2} + \gamma\left|u\right|^2 u = 0$$

$$u = \sqrt{2}A\cosh^{-1}\left[A\sqrt{\gamma}z\right]\exp i\gamma A^2 t$$

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial z^2} - \gamma_1\left|u\right|u + \gamma_2\left|u\right|^2 u = 0$$

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial z^2} + \gamma_1\left|u\right|^2 u + \gamma_2\left|u\right|^4 u = 0$$

$$i\frac{\partial u}{\partial t} + D\frac{\partial^2 u}{\partial x^2} - \gamma_1 |u| u + \gamma_2 |u|^2 u = 0,$$

una variante de la ecuación no linear de Schrodinger, ha encontrado aplicaciones en muy diversas áreas de la física, como son

- Óptica no lineal.
- Condensados de Bose -Einstein.
- Transmisión de señales en fibras ópticas.

Esta ecuación admite soluciones tipo solitón, tanto regulares como racionales.
Entre las racionales está

$$u(x,t) = \frac{1}{M + Nx^2},$$

con

$$M = \frac{3\gamma_2}{4\gamma_1} \text{ y } N = \frac{\gamma_1}{6D},$$

lo que implica que

$$u(x,t) = \frac{1}{M + N(x - Vt)^2} e^{i(Rx - St)},$$

con

$$R = \frac{V}{2D}, \; S = \frac{V^2}{4D} \text{ y } V \text{ una constante arbitraria}$$

es también una solución.

Entre las regulares está

$$u = \frac{A_0}{B_0 \cosh\left[C_0\left(x - V_0 t\right)\right] - 1} e^{i(k_0 x - \mu_0 t)}$$

con velocidad $V_0$ constante arbitraria y

$$\begin{aligned} k_0 &= \frac{V_0}{2D}, \\ A_0 &= \frac{3D}{\gamma_1}\left(k_0^2 - \frac{\mu_0}{D}\right), \\ B_0 &= \sqrt{\frac{9D\gamma_2}{2\gamma_1^2}\left(k_0^2 - \frac{\mu_0}{D}\right) + 1}, \\ C_0 &= \sqrt{k_0^2 - \frac{\mu_0}{D}} \end{aligned}$$

y con la condición

$$k_0^2 - \frac{\mu_0}{D} > 0.$$

¿Qué sucede si $\gamma_1 = \Gamma_1[1 + \epsilon \text{sen}(\omega t)]$?

Para investigar esto utilizaremos una aproximación variacional.

- Ecuación en derivadas parciales generada por una densidad lagrangiana.
- Una solución exacta.
- Una propuesta de solución cercana a la exacta, pero con algunas funciones del "tiempo" por determinar.
- Se substituye la propuesta en la densidad lagrangiana y se integra sobre las variables "espaciales".
- Se obtiene una lagrangiana efectiva en la que las funciones del "tiempo" aparecen como coordenadas.
- Las ecuaciones de Euler Lagrange determinan las funciones.

Comencemos con el caso en que buscamos soluciones cercanas al solitón racional. En este caso cuando $\gamma_1$ es constante tenemos que:

- Ecuación

$$i\frac{\partial u}{\partial t}+D\frac{\partial^2 u}{\partial x^2}-\gamma_1\left|u\right|u+\gamma_2\left|u\right|^2 u=0.$$

- Densidad lagrangiana

$$\mathcal{L}=i(u^* u_t-uu_t^*)-\frac{4}{3}\gamma_1\left|u\right|^3+\gamma_2\left|u\right|^4-2D\left|u_x\right|^2.$$

- Solución exacta

$$u\left(x,t\right)=\frac{1}{M+N\left(x-Vt\right)^2}e^{i(Rx-St)},$$

con

$$R=\frac{V}{2D},\quad S=\frac{V^2}{4D},\quad M=\frac{3\gamma_2}{4\gamma_1},\quad N=\frac{\gamma_1}{6D}$$

y $V$ arbitraria.

- Propuesta

$$u(x,t) = \frac{1}{f(t)+g(t)x^2} \exp i \left[h(t) + b(t)x^2\right].$$

- Lagrangiana efectiva

$$L = -\frac{\pi \dot{h}}{g^{\frac{1}{2}} f^{\frac{2}{3}}} - \frac{\pi \left[\dot{b} + 4Db^2\right]}{g^{\frac{3}{2}} f^{\frac{1}{2}}} - \frac{\pi \gamma_1}{2 g^{\frac{1}{2}} f^{\frac{5}{2}}} + \frac{5\pi\gamma_2}{16 g^{\frac{1}{2}} f^{\frac{7}{2}}} - \frac{\pi D g^{\frac{1}{2}}}{2 f^{\frac{5}{2}}}.$$

- Ecuaciones de Euler Lagrange

$$\frac{dg}{dt} = -6Dgb,$$

$$\frac{df}{dt} = 2Dbf,$$

$$\frac{db}{dt} = \frac{\gamma_1}{8}\frac{g}{f^2} - \frac{5\gamma}{32}\frac{g}{f^3} + \frac{D}{2}\frac{g^2}{f^2} - 4Db^2,$$

$$\frac{dh}{dt} = \frac{25}{32}\frac{\gamma_2}{f^2} - \frac{7}{8}\frac{\gamma_1}{f} - \frac{Dg}{f}.$$

Tenemos que $gf^3$ es una integral de movimiento que llamaremos $C$. El sistema se reduce a

$$\frac{dg}{dt} = -6Dgb,$$

$$\frac{db}{dt} = \left(\frac{1}{8C^{\frac{2}{3}}}\right)\gamma_1 g^{\frac{5}{3}} - \left(\frac{5}{32C}\right)\gamma_2 g^2 + \left(\frac{D}{2C^{\frac{2}{3}}}\right) g^{\frac{8}{3}} - 4Db^2.$$

Estas ecuaciones se pueden derivar de la lagrangiana

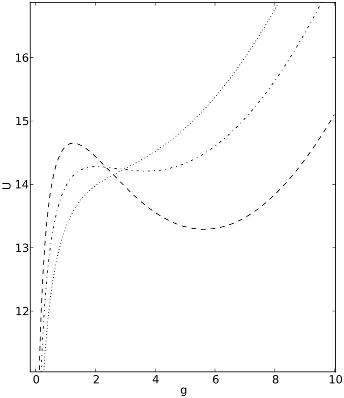
$$L_{var} = g^{-\frac{10}{3}}\dot{g}^2 - 9D\left[\frac{1}{2}C^{-\frac{2}{3}}\gamma_1 g^{\frac{1}{3}} - \frac{5}{16}C^{-1}\gamma_2 g^{\frac{2}{3}} + \frac{1}{2}DC^{-\frac{2}{2}}g^{\frac{4}{3}}\right]$$

o, al definir $p = \partial L_{var}/\partial \dot{g}$, de la hamiltoniana

$$H_{var} = \frac{1}{4}g^{\frac{10}{3}}p^2 + \frac{9}{2}D\left[C^{-\frac{2}{3}}\gamma_1 g^{\frac{1}{3}} - \frac{5}{8}C^{-1}\gamma_2 g^{\frac{2}{3}} + DC^{-\frac{2}{3}}g^{\frac{4}{3}}\right].$$

Esto lo podemos ver como un movimiento unidimensional en un potencial efectivo

$$U_{ef}(g) = \frac{9}{2}D\left[C^{-\frac{2}{3}}\gamma_1 g^{\frac{1}{3}} - \frac{5}{8}C^{-1}\gamma_2 g^{\frac{2}{3}} + DC^{-\frac{2}{3}}g^{\frac{4}{3}}\right]$$

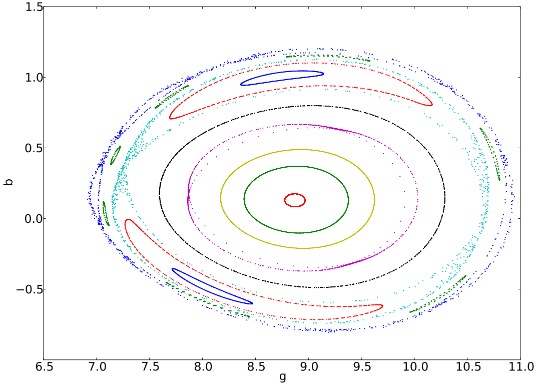



Potencial efectivo para $D = 0.5$, $\gamma_1 = 11$, $\gamma_2 = 9$ y diferentes valores de $C \equiv g(0)f^3(0)$. Punto y raya $f(0) = M$ y $g(0) = N$, solución racional. Rayas $f(0) = 0.95M$ y $g(0) = N$, solución periódica que representa un solitón oscilante estable. Puntos $f(0) = 1.05M$ y $g(0) = N$, no hay solución periódica estable y $g$ tiende a cero: el solitón se destruye.

Cuando $\gamma_1$ no es constante y la tenemos como $\gamma_1(t) = \Gamma_1(1 + \epsilon \mathrm{sen}(\omega t))$ podemos ver que todo el análisis variacional que hicimos sigue siendo válido excepto por que la hamiltoniana (o el potencial efectivo) adquiere un término adicional dependiente del tiempo

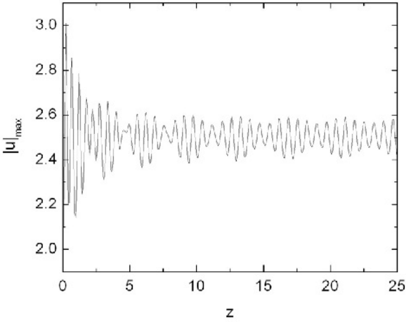
$$\epsilon \frac{9}{2} D C^{-\frac{2}{3}} \Gamma_1 g^{\frac{1}{3}} \mathrm{sen}(\omega t)$$

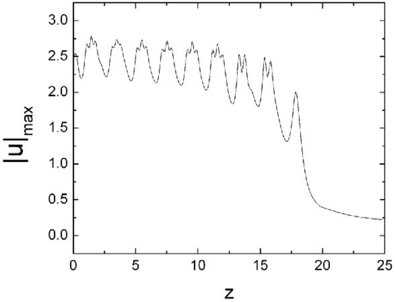
el que, para $\epsilon$ pequeña, podemos ver como una perturbación a la hamiltoniana integrable $H_{var}$. Podemos entonces aplicar el teorema de KAM y hacer ver que el sistema tendrá, para $\epsilon$ suficientemente pequeña, soluciones periódicas, cuasiperiódicas (con respecto a la frecuencia $\omega$) y caóticas.




Mapeo de Poincaré en espacio $(g, b)$ para $\Gamma_1 = 11$, $\gamma_2 = 9$, $D = 0.5$, $\epsilon = 0.2$, $\omega = 1.766537$ y diez condiciones iniciales $(g(0), b(0))$: $(8.8, 0.1)$, $(8.4, 0.1)$, $(8.4, -0.1)$, $(7.899335, 0)$, $(8.4, -0.4)$, $(7.4148, 0)$, $(7.4148, -0.075)$, $(8.4, -0.575)$, $(7.1, 0)$ y $(7.02, 0)$.

Soluciones numéricas de la PDF con condiciones iniciales cercanas al solitón racional.

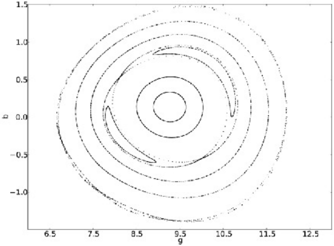


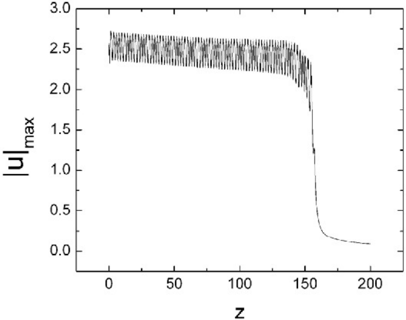




$M_0 = 0.85M$ y $N_0 = 0.7N$. $\Gamma_1 = 11$, $\gamma_2 = 9$, $D = 0.5$, $\epsilon = 0.04$, $\omega = 8.85$

$B = 0.1$, $N_0 = 8.8$, $M_0 = (C/N_0)^{1/3}$ y $C = N\ (0.85M)^3$.

$\Gamma_1 = 11$, $\gamma_2 = 9$, $D = 0.5$, $\epsilon = 0.2$, $\omega = 3.126770$




$\Gamma_1 = 11,\ \gamma_2 = 9,\ D = 0.5,\ \epsilon = 0.14,\ \omega = 3$

$\Gamma_1 = 11,\ \gamma_2 = 9,\ D = 0.5,\ \epsilon = 0.2,\ \omega = 1.766537$

El caso de solitones hiperbólicos para la ecuación

$$i\frac{\partial u}{\partial t} + D\frac{\partial^2 u}{\partial x^2} - \gamma_1 |u|\, u + \gamma_2 |u|^2\, u = 0$$

con $\gamma_1$ constante $(D, \gamma_1, \gamma_2 > 0)$ tiene la solución exacta

$$u = \frac{A_0}{B_0 \cosh\left[C_0\left(x - V_0 t\right)\right] - 1} e^{i(k_0 x - \mu_0 t)},$$

con

$$\begin{aligned} k_0 &= \frac{V_0}{2D}, \\ A_0 &= \frac{3D}{\gamma_1}\left(k_0^2 - \frac{\mu_0}{D}\right), \\ B_0 &= \sqrt{\frac{9D\,\gamma_2}{2\,\gamma_1^2}\left(k_0^2 - \frac{\mu_0}{D}\right) + 1}, \\ C_0 &= \sqrt{k_0^2 - \frac{\mu_0}{D}}, \end{aligned}$$

$$k_0 - \frac{\mu_0}{D} > 0.$$

Hacemos un estudio similar para este caso.
La propuesta para la aproximación variacional que hacemos es

$$u(x,t) = \frac{A(t)}{\cosh\left(x/w(t)\right)} \exp i\left[h(t) + b(t)x^2\right]$$

Al substituir en la densidad lagrangiana e integrar obtenemos la lagrangiana

$$L_{var} = -4A^2\dot{h} - \frac{1}{3}\pi^2 w^3 A^2 \dot{b} - \frac{4}{3}\pi^2 D w^3 b^2 - \frac{2}{3}\pi\gamma_1 w A^3 + \frac{4}{3}\gamma_2 w A^4 - \frac{4}{3}\frac{DA^2}{w}A^2$$

y las ecuaciones de Euler Lagrange

$$\frac{d}{dt}\left(wA^2\right) = 0, \qquad \frac{dw}{dt} = 4Dbw$$

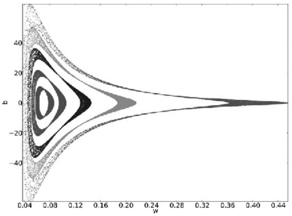
$$-12w^2\frac{dh}{dt} - \pi^2 w^4 \frac{db}{dt} - 4\pi^2 D w^4 b^2$$
$$-3\pi\gamma_1 w^2 A + 8\gamma_2 w^2 A^2 - 4D = 0,$$

$$12w^2\frac{dh}{dt} + 3\pi^2 w^4 \frac{db}{dt} + 12\pi^2 D w^4 b^2$$
$$+2\pi\gamma_1 w^2 A - 4\gamma_2 w^2 A^2 - 4D = 0.$$

La primera ecuación nos proporciona una integral de movimiento y al sumar la segunda con la tercera una ecuación de segundo orden en $w$

$$\frac{d^2w}{dt^2} = -\frac{d}{dw}\left(\frac{8D^2}{\pi^2 w^2} - \frac{8D\gamma_2 c_0}{\pi^2 w} + \frac{4D\gamma_1\sqrt{c_0}}{\pi\sqrt{w}}\right).$$

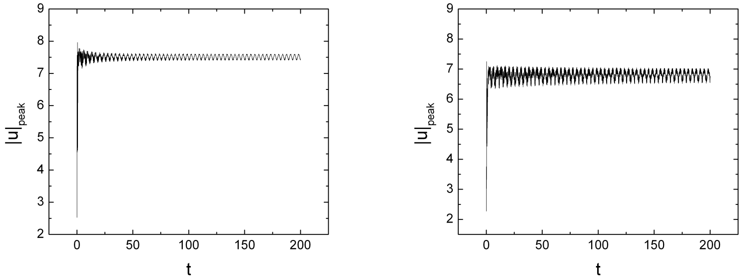
Esto puede verse de manera hamiltoniana con un potencial efectivo. Cuando $\gamma_1$ varia con el tiempo se aplica el teorema de KAM y tendremos soluciones periódicas, cuasiperiódicas y caóticas.




$\Gamma_1 = 11$, $\gamma_2 = 9$, $D = 0.5$, $\epsilon = 0.2$, $\omega = 1.766537$ y $c_0 = 1.956530288$.

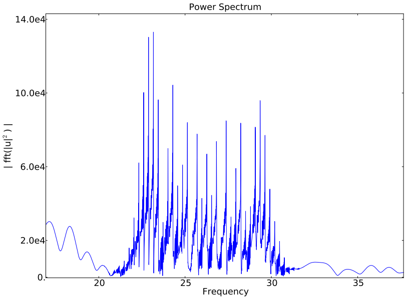
Resultados numéricos para el caso hiperbólico con condiciones iniciales de la forma

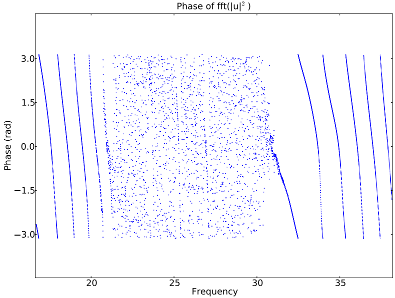
$$u(x, t = 0) = \frac{a}{b\cosh(cx) - 1}.$$



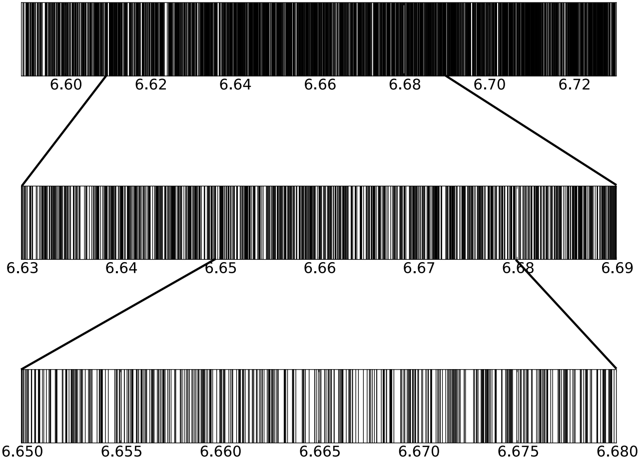

$\Gamma_1 = 11$, $\gamma_2 = 9$, $D = 0.5$, $\epsilon = 0.2$, $\omega = 1.766537$, con $a = 6\sqrt{2}\xi$, $b = \sqrt{19}$, $c = \sqrt{2}$, $\xi = 1.1$ y $\xi = 0.9$

## Análisis de la solución numérica para $\xi = 0.9$. Espectro de potencias de $|u(x=0,t)|^2$

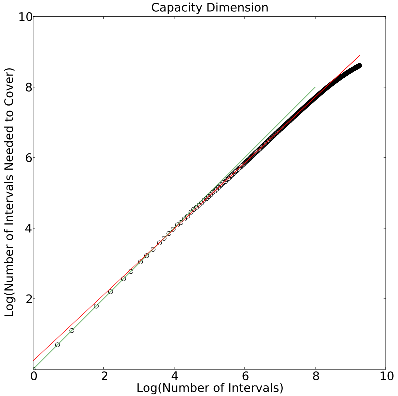


Conjunto de intersecciones de $u(x = 0, t)$ con el eje real, 10500 intersecciones.



# Dimensión de capacidad del conjunto de intersecciones.




$N_\epsilon$ entre 145 y 2799 lineal con pendiente 0.9347 y una desviación cuadrática media de $3 \times 10^{-5}$.
La dimensión de correlación de 20000 puntos de la trayectoria da 1.875 y desviación cuadrática menor de $10^{-4}$.

# CONCLUSIONES

- Al introducir la posibilidad de un manejo no lineal en ecuaciones con soluciones tipo solitón se observan diferencias entre los casos hiperbólicos y racional. Los primeros presentan pulsos robustos, mientras los segundos no los tienen.
- En el caso hiperbólico se abre la posibilidad de tener “solitones caóticos”.
- Las aproximaciones variacionales predicen soluciones caóticas en ambos casos, pero solo en el hiperbólico se ven muchas más posibilidades de que esas predicciones sean reales.
- Se abre la posibilidad de estudiar PDE con comportamiento caótico.