The end of paring-based cryptography using small characteristic finite fields

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In joint work with

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2. Small-Characteristic Pairings
3. Security of Small-Characteristic Pairings
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Pairing-Based Cryptography

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- Immediate property: for any integer k,

$$\hat{e}(kQ, R) = \hat{e}(Q, R)^k = \hat{e}(Q, kR).$$

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- **Short digital signatures**
  - Boneh–Lynn–Shacham, 2001.
  - Zang–Safavi-Naini–Susilo, 2004.

Small-Characteristic Pairings

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Most common pairing maps:

- Weil pairings.
- Tate pairings and modifications (Eta, Ate, ...).

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- The $k = 6$ pairings derived from supersingular elliptic curves over $\mathbb{F}_{3^n}$:
  - $Y^2 = X^3 - X + 1$; and
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  - $Y^2 = X^3 - X + 1$; and
  - $Y^2 = X^3 - X - 1$.
- The $k = 12$ pairing derived from supersingular gen.-2 curves over $\mathbb{F}_{2^n}$:
  - $Y^2 + Y = X^5 + X^3$; and
  - $Y^2 + Y = X^5 + X^3 + 1$.

Security of Small-Characteristic Pairings (Prior to 2013)

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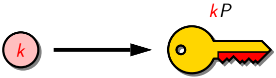

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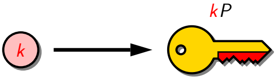

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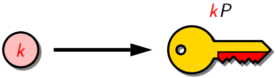

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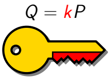

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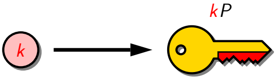

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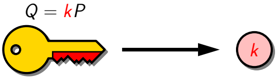

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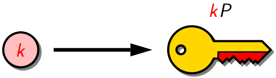

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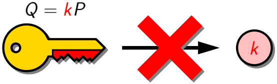

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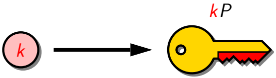

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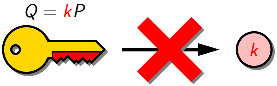

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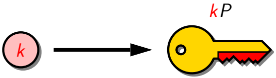

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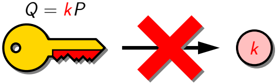

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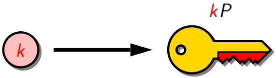

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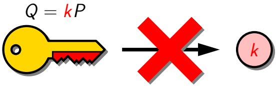

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DLP: given $h \in \mathbb{G}_T$, find $0 \leq i < r$ such that $h = g^i$.

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However, efficient problem reductions exist:

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- For pairing-based cryptography over supersingular curves:
  - The embedding degree is relatively small ($k = 4$, 6, or 12).
  - So, the finite field $\mathbb{F}_{q^k}$ (containing $\mathbb{G}_T$) is not very large.

DLP algorithms for small-characteristic fields $\mathbb{F}_Q$

- Subexponential running time, for $0 < \alpha < 1$ and $c > 0$, at input Q:

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Table: Security of small-characteristic parings as in 2012 (DLP in $\mathbb{F}_{p^{kn}}$)

Underlying field ($\mathbb{F}p^n$)	$\mathbb{F}_{2^n}$	$\mathbb{F}_{3^n}$	$\mathbb{F}_{2^n}$
Embedding degree (k)	4	6	12
Lower security ($\approx 2^{64}$)	$n = 239$	$n = 97$	$n = 127$
Medium security ($\approx 2^{80}$)	$n = 373$	$n = 163$	$n = 163$
Standard security ($\approx 2^{128}$)	$n = 1223$	$n = 509$	$n = 367$
Higher security ($\approx 2^{192}$)	$n = 3041$	$n = 1429$	$n \approx 983$

Joux-Lercier algorithm for $\mathbb{F}_Q = \mathbb{F}_{q^n}$

- In 2006, Joux and Lercier [JL06] presented an algorithm with running time $L_Q[\frac{1}{3}, 1.442]$ when q and n are 'balanced'

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Extension Field $\mathbb{F}_{3^{6 \cdot n}}$	$n = 97$	$n = 163$	$n = 509$
Security level	$2^{52.79}$	$2^{68.17}$	$2^{111.35}$

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- Later in 2012, Joux [Joux12] introduced a technique that improved the [JL06] algorithm to $L_Q[\frac{1}{3}, 0.961]$.

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- **Feb, Apr 2013** - Göloğlu-Granger-McGuire-Zumbrägel:
  - presented ideas somewhat similar to Joux's.

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First Contributions

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- Descent stage: $\log_g h$ is expressed as a linear combination of logs of elements in the factor base using classical methods and a new descent method (based on solving multivariate bilinear equations).

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- In this case, we say that we expressed $\log_g f$ as a linear combination of logarithms of polynomials of degree at most m.

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$X^{136} + \ldots$

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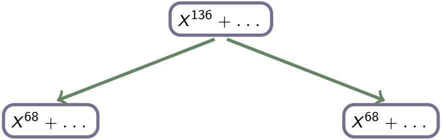

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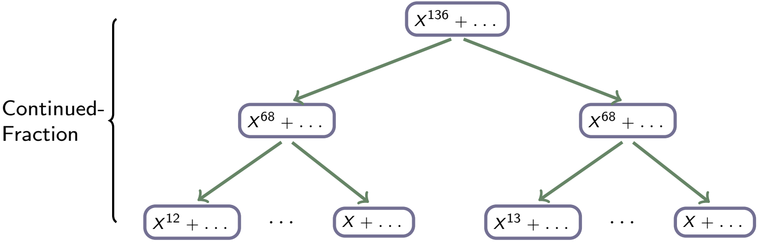

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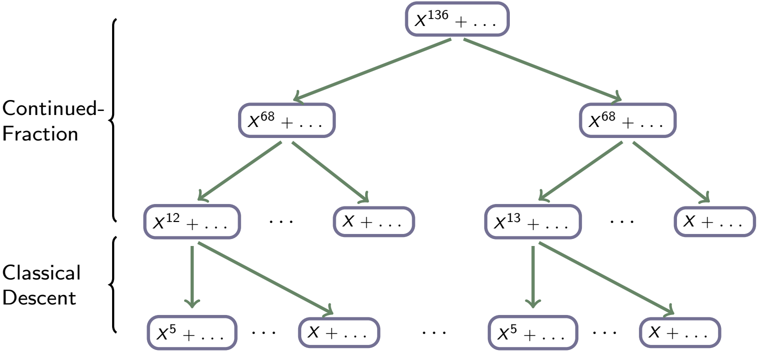

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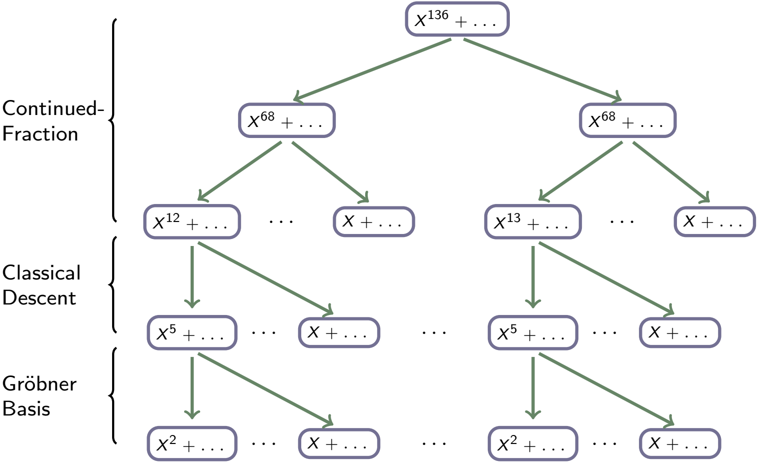

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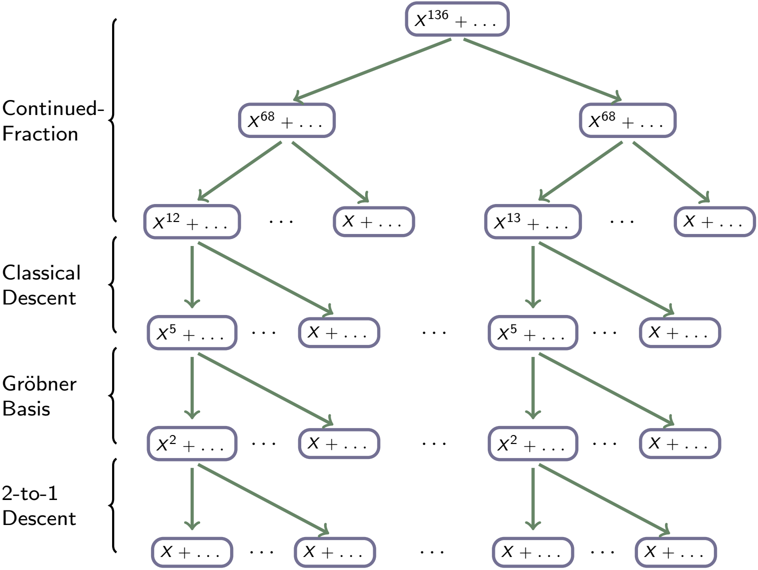

More Improvements

Practical improvements

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The 509's Computations

Latest computations on $\mathbb{F}_{3^{6\cdot 509}}$

July 18 2016, A.-Canales-Cruz-Menezes-Oliveira-Rivera-Rodríguez

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Running-time

Computation stage	CPU time (years)		CPU frequency (GHz)
Finding logarithms of quadratic polynomials			
Relation generation	0.01	(CS Dept.)	3.20
Linear algebra	0.50	(CS Dept.)	2.40
Finding logarithms of cubic polynomials			
Relation generation	0.15	(CS Dept.)	3.20
Linear algebra	43.88	(ABACUS)	2.60
Finding logarithms of quartic polynomials			
Relation generation	4.07	(CS Dept.)	2.60
Linear algebra	96.02	(ABACUS)	2.60
Descent			
Continued-fractions (254 to 40)	51.71	(CS Dept.)	2.87
Classical (40 to 21)	9.99	(CS Dept., U Wat.)	2.66
Classical (21 to 15)	10.24	(CS Dept., U Wat.)	2.66
Gröbner bases (15 to 4)	6.27	(CS Dept., U Wat.)	3.00
Total CPU time (years)	222.81		

Table: CPU times of each stage of the discrete logarithm computation in $\mathbb{F}_{3^{6\cdot 509}}$.

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  - Logarithms in the VM are read by Magma and those in the HD from some C-codes called from Magma. The System V shared memory is used for the Magma-C for the interprocess communication.

DLP at The 192-bit Security Level

Guillevic's descent method (July 2016)

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- In our case, we choose n' so that $3^{6n'-3n} \gg q^{n'}/N_q(m,n')$, where $N_q(m,n')$ denotes the number of monic m-smooth degree-n' polynomials in $\mathbb{F}_q[X]$.

Discrete logarithms in $\mathbb{F}_{3^{6\cdot 1429}}$

- $E : Y^2 = X^3 - X - 1$ a supersingular elliptic curve over $\mathbb{F}_3$. $|E(\mathbb{F}_{3^{1429}})| = cr$, where $c = 7622150170693$ and $r = (3^{1429} - 3^{715} + 1)/c$, a 2223-bit prime.

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Finding logarithms of quadratic polynomials	
Degree 1 and 2	$2^{50.7}$
Degree 3	$2^{56.9}$
Degree 4 (36/728)	$2^{56.3}$
Descent	
Guillevic (1428 to 71)	$2^{62.4}$
Classical (71 to 32)	$2^{61.8}$
Classical (31 to $\{1, \ldots, 16, 18, 20, 22, 24, 28, 32\}$)	$2^{59.2}$
Small degree ($\{5, \ldots, 16, 18, 20, 22, 24, 28, 32\}$ to 4)	$2^{60.0}$
Total cost	$2^{63.4}$

Feasibility of computing discrete logarithms in $\mathbb{F}_{3^{6\cdot 1429}}$

- We assume that we have access to a 9000-core cluster $\mathcal{A}$, where each core has access to 16 gigabytes of shared RAM, such as ABACUS-Cinvestav.

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Computation	Cluster	# cores	# days
Degree-3	$\mathcal{A}$	5824	2
Degree-4	$\mathcal{A}$	9000	1
Guillevic descent	$\mathcal{A}$	9000	59
First classical descent	$\mathcal{A}$	9000	39
Second classical descent	$\mathcal{A}$	9000	7
Small degree descent	$\mathcal{B}$	1500	65
Total time			173

Table: Estimated calendar time for computing discrete logarithms in $\mathbb{F}_{3^{6\cdot 1429}}$ using clusters $\mathcal{A}$ and $\mathcal{B}$.

Open problem

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Muchas Gracias

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The continued-fraction descent

Recall that we want to compute $\log_g h$ and assume that n is odd and $\deg h = n - 1$.

For a chosen $m < n - 1$, we want to express $\log_g h$ as a linear combination of logarithms of polynomials of degree at most m.

- (1) Multiply h by a random power of g to get $h' = g^i * h$.
- (2) Use the extended Euclidean algorithm to express h' in the form

$$w_2 \cdot h' + v \cdot I_X = w_1 \quad \text{where} \quad \deg w_1 = \deg w_2 = \frac{n-1}{2}.$$

Repeat (1)-(2) until w_1 and w_2 are m-smooth.

In $\mathbb{F}_{3^{6 \cdot 137}}$, for $m = 13$, the total running time of the continued-fraction step is 22 CPU hours.

The Gröbner bases descent

Let $f \in \mathbb{F}_{q^d}[X]$ of degree D, and let $m = \lceil (D+1)/2 \rceil$.

We want 2 polynomials k_1 and $k_2 \in \mathbb{F}_{q^d}[X]$ of degree d such that $f \mid G$,

$$\text{where} \quad G = k_1 \widetilde{k}_2 - \widetilde{k}_1 k_2 \pmod{I_X},$$

with $\widetilde{k}_i(X) = \overline{h}_1^m \cdot \overline{k}_i \left(\frac{\overline{h}_0}{\overline{h}_1}\right)$ and $\widetilde{k}_i(X) = \overline{h}_1^m \cdot \overline{k}_i \left(\frac{\overline{h}_0}{\overline{h}_1}\right)$.
We then have

$$G^q \equiv \overline{h}_1^{mq} \cdot k_2 \cdot \prod_{\lambda \in \mathbb{F}_q} (k_1 - \lambda k_2) \pmod{I_X}.$$

as can be seen by making the substitution $Y \mapsto k_1/k_2$ into the systematic equation $Y^q - Y = \prod_{\lambda \in \mathbb{F}_q} (Y - \lambda)$ and clearing denominators.

If $3m < n$, then $G = k_1 \widetilde{k}_2 - \widetilde{k}_1 k_2$, since $k1\widetilde{k}_2 - \widetilde{k}_1 k2$ has degree $3m$ and so $G(X) = f(X)R(X)$ for some $R \in \mathbb{F}_{q^d}[X]$ with $\deg R = 3m - D$.