Encajes de espacios de quregistros en grupos especiales lineales

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Contenido

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- Para un cuerpo negro (T) la densidad espectral del campo de radiación:

$$\rho(\omega)_{Cl} = \frac{\omega^2}{\pi^2 c^3} kT; \quad \rho(\omega)_Q = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/kT} - 1} \rightsquigarrow u = \frac{\pi^2 k^4}{15 c^3 \hbar^3} T^4$$

Ley de Stefan-Boltzmann (1884)

$$u = \frac{4\sigma}{c} T^4$$

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Interpretación de Copenhage

- Describe la evolución temporal de partículas subatómica masivas de naturaleza ondulatoria y no relativista (onda-partícula).

$$i\hbar\frac{\partial|\psi\rangle}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2|\psi\rangle + V|\psi\rangle, \quad \text{ecuación de Schrödinger}$$

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- Observables $(\mathbf{x}, m\mathbf{v}, E, S_z, \mu, \ldots) \rightsquigarrow \quad A : \mathbb{H} \to \mathbb{H} \rightsquigarrow \mathbb{H} = \mathbb{H}^\dagger$ $(\mathbb{H}, \quad dim(\mathbb{H})_\mathbb{C} = n)$.

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- $\mathbb{H} \backsimeq \mathbb{H}^* \rightsquigarrow \{|x\rangle\} \quad \leftrightarrow \quad S_\mathbb{H} = \{x \in \mathbb{H}| \ \langle x|x\rangle = 1\} = \{\text{ estados }\}$

Mecánica cuántica

Interpretación de Copenhage

- DE $\rightsquigarrow$ $A = \sum_{i=0}^{n-1} a_i \, \pi_{x_i}$, con $\pi_{x_i} = x_i \, x_i^H$, $x_i \in S_{\mathbb{H}}$
- $\{x_i\}, \{a_i\}$, $i = 0, \dots, n-1$ $\rightsquigarrow$ $\text{eigenestados}_A, \text{eigenvalores}_A$, $\langle x_i | x_j \rangle = \delta_{ij}$.

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- $\{x_i\}, \{a_i\}$, $i = 0, \ldots, n-1$ $\rightsquigarrow$ eigenestados$_A$, eigenvalores$_A$, $\langle x_i | x_j \rangle = \delta_{ij}$.
- Una medición $\mu(A, x)$ de A en $x \in S_{\mathbb{H}}$, $x = \sum_{i=0}^{n-1} a_i \, x_i$,

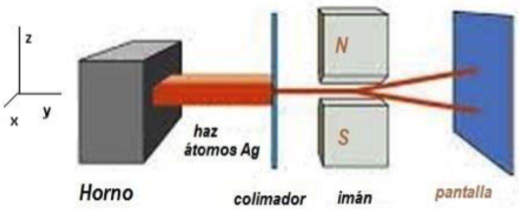
$$\Pr\left(\mu(A, x) = a_i\right) = |a_i|^2.$$

- En MQ, el acto de medición altera en forma irreversible la situación del sistema que prevalecía antes de medirse (colapso de la función de onda).

Mecánica cuántica experimental

Dispositivo Stern-Gerlach

$Ag \rightsquigarrow \quad 1n,\ 47e \quad \rightsquigarrow \quad 46e - \circledS,\ \mathbf{J}_{46} = 0 \rightsquigarrow \quad \mathbf{J}_{47} := \mu \propto \mathbf{S}_{47vo} := \mathbf{S}$



Dispositivo Stern-Gerlach

espectro discreto

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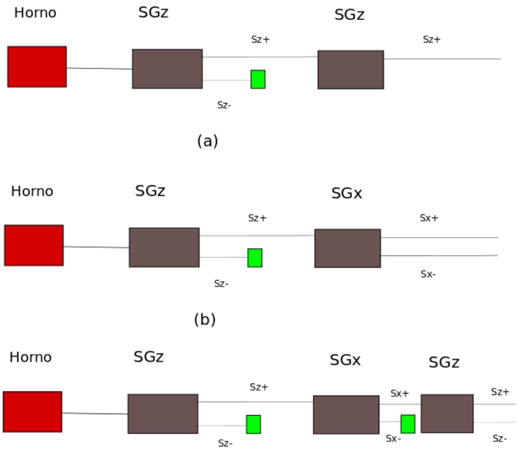
- $$F_z = \frac{\partial}{\partial z}(\mu \cdot B) \backsimeq \mu_z \frac{\partial B_z}{\partial z}.$$
- SG detecta μ_z
- Si tuviera un comportamiento clásico $\rightsquigarrow$ continuo $\mu_z \in [-|\mu|, |\mu|]$.
- $$S_z^+ = \frac{\hbar}{2} \leftrightsquigarrow |+\rangle, \quad S_z^- = -\frac{\hbar}{2} \leftrightsquigarrow |-\rangle.$$

Curiosidades de la naturaleza cuántica- perdida de información

Una serie de dispositivos de Stern-Gerlach



Mecánica cuántica

Sistemas compuesto y enredamiento

- Sistemas compuestos o multipartitas $\rightsquigarrow$ estructura interna $\rightsquigarrow$ 2 o más subsistemas $\mathbb{H}_n = \rightsquigarrow \quad \mathbb{H} \otimes \cdots \otimes \mathbb{H}$.

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- (sí,no)- separable $x = c_0 \otimes \cdots \otimes c_n$ (no,sí)- enredamiento
- $\exists$ criterios de separabilidad y medidas del enredamiento

Cómputo cuántico

- Qubits $\leftrightsquigarrow$ combinaciones normalizadas de $|+\rangle$ y $|-\rangle$
- Quregistros $\leftrightsquigarrow$ elementos en $\mathbb{H}_n$
- compuertas cuánticas$\leftrightsquigarrow$ operadores $\mathbb{H}_n \to \mathbb{H}_n$

Preliminaries

- Let $S_2 = \{\mathbf{x} = (x_0, x_1) \in \mathbb{H}_1 \mid |x_0|^2 + |x_1|^2 = 1\}$ the unit sphere of $\mathbb{H}_1 = \mathbb{C}^2$.

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- $\{\text{Qubits}\} = S_2 \quad \text{for} \quad \mathbb{H}_1$
- $\exists \quad \Psi_1 : S_2 \to \mathrm{SU}(2) \ , \ \mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \mapsto \Psi_1(\mathbf{x}) = \begin{bmatrix} x_0 & -\overline{x_1} \\ x_1 & \overline{x_0} \end{bmatrix}$

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- S_2 inherits the algebraic structure of $\mathrm{SU}(2)$

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- $\forall U \in \mathrm{U}(2): \left[\Psi_1 \circ U = U \circ \Psi_1 \iff U \in \mathrm{SU}(2)\right]$,

  where $\Psi_1 : S_2 \to \mathrm{SU}(2)$

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Contenido

Embedding S_{2^2} into $\mathrm{SL}(2^2)$

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- (2-quregisters) $\subset S_{2^2}$ of $\mathbb{H}_2 = \mathbb{H}_1 \otimes \mathbb{H}_1$
- 2-quregister $\mathbf{x} \in S_{2^2}$ is separable if $\exists\ \mathbf{c}_0, \mathbf{c}_1 \in S_2$ such that

$$\mathbf{x} = \mathbf{c}_0 \otimes \mathbf{c}_1$$

Case of 2-quregisters

- Proposition: A 2-quregister $\mathbf{x} = [x_0 \;\; x_1 \;\; x_2 \;\; x_3]^T \in S_{2^2}$ is separable $\iff x_0 x_3 = x_1 x_2$.

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- For $j \in \{0, 1, 2, 3\}$, let $C_j = \{\mathbf{x} \in S_{2^2} |\; x_j \neq 0\}$

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- For $j \in \{0, 1, 2, 3\}$, let $C_j = \{\mathbf{x} \in S_{2^2} | \ x_j \neq 0\}$
- Each C_j is a chart in S_{2^2} (DM)

Case of 2-quregisters

- Proposition: If a 2-quregister $\mathbf{x} = [x_0 \;\; x_1 \;\; x_2 \;\; x_3]^T \in S_{2^2}$ is separable, then it can be tensor splitted as $\mathbf{x} = \mathbf{c}_0 \otimes \mathbf{c}_1$, where the qubits $\mathbf{c}_0, \mathbf{c}_1$ are determined according to the following rules:

$$\mathbf{x} \in C_0 \implies \mathbf{c}_0 = \frac{1}{r_{02}} \begin{bmatrix} x_0 \\ x_2 \end{bmatrix} \wedge \mathbf{c}_1 = r_{02} \begin{bmatrix} 1 \\ \frac{x_1}{x_0} \end{bmatrix}$$

$$\mathbf{x} \in C_1 \implies \mathbf{c}_0 = \frac{1}{r_{13}} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \wedge \mathbf{c}_1 = r_{13} \begin{bmatrix} \frac{x_0}{x_1} \\ 1 \end{bmatrix}$$

Case of 2-quregisters

-

$$\mathbf{x} \in C_2 \implies \mathbf{c}_0 = \frac{1}{r_{02}} \begin{bmatrix} x_0 \\ x_2 \end{bmatrix} \wedge \mathbf{c}_1 = r_{02} \begin{bmatrix} 1 \\ \frac{x_3}{x_2} \end{bmatrix},$$

$$\mathbf{x} \in C_3 \implies \mathbf{c}_0 = \frac{1}{r_{13}} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \wedge \mathbf{c}_1 = r_{13} \begin{bmatrix} \frac{x_2}{x_3} \\ 1 \end{bmatrix}.$$

with $r_{02} = \sqrt{|x_0|^2 + |x_2|^2}$ and $r_{13} = \sqrt{|x_1|^2 + |x_3|^2}$

Case of 2-quregisters

- Moreover, since $\mathbf{x} = \mathbf{c}_0 \otimes \mathbf{c}_1$, for any unit $u \in \mathbb{C}$, $\mathbf{x} = (u^{-1}\mathbf{c}_0) \otimes (u\mathbf{c}_1)$ is another tensor split

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- For $k \in \{0, 1, 2, 3\}$ and a unit $u \in \mathbb{C}$, let $\Phi_{2ku} : C_k \to \mathbb{C}^{2^2 \times 2^2}$, for $\mathbf{x} \in C_k$

$$\Phi_{20u}(\mathbf{x}) \equiv \begin{bmatrix} x_0 & -u^{-2}\xi(x_0)^2\overline{x_1} & -u^2\overline{x_2} & \overline{x_3} \\ x_1 & u^{-2}x_0 & -u^2\xi(x_2)^{-2}x_3 & -\overline{x_2} \\ x_2 & -u^{-2}\xi\left(\frac{x_1}{x_0}\right)^{-2}x_3 & u^2\overline{x_0} & -\overline{x_1} \\ x_3 & u^{-2}x_2 & u^2\xi(x_0)^{-2}x_1 & \overline{x_0} \end{bmatrix}$$

Case of 2-quregisters

- $$\Phi_{21u}(\mathbf{x}) = \begin{bmatrix} x_0 & -u^{-2}x_1 & -u^2\xi(x_3)^{-2}x_2 & \overline{x_3} \\ x_1 & u^{-2}\xi(x_1)^2\overline{x_0} & -u^2\overline{x_3} & -\overline{x_2} \\ x_2 & -u^{-2}x_3 & u^2\xi(x_1)^{-2}x_0 & -\overline{x_1} \\ x_3 & u^{-2}\xi\left(\frac{x_0}{x_1}\right)^{-2}x_2 & u^2\overline{x_1} & \overline{x_0} \end{bmatrix}$$

  where $\xi : \mathbb{C} \to \mathbb{C}$,

  $$z \mapsto \xi(z) = \begin{cases} \frac{z}{|z|} & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

Case of 2-quregisters

- For any separable 2-quregister $x \in Sp_2$

$$\Phi_{2ku}(\mathbf{x})^H \Phi_{2ku}(\mathbf{x}) = \mathbb{I}_{2^2}$$

for any $k \in \{0, 1, 2, 3\}$

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- For any separable 2-quregister $x \in Sp_2$

$$\Phi_{2ku}(\mathbf{x})^H \Phi_{2ku}(\mathbf{x}) = \mathbb{I}_{2^2}$$

for any $k \in \{0, 1, 2, 3\}$

- Consequently for $k \in \{0, 1, 2, 3\}$, the map $\Phi_{2ku} : C_k \to \mathbb{C}^{2^2 \times 2^2}$ determines

$$C_k \cap Sp_2 \hookrightarrow \mathrm{SU}(2^2)$$

Canonical Basis

- Example
  The i-th vector $\mathbf{e}_i = [\delta_{ij}]_{j=0}^{3}$ in the canonical basis of $\mathbb{H}_2$ is a separable 2-quregister:

$$\mathbf{e}_{2i_1+i_0} = \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_0}$$

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We have $\mathbf{e}_i \in C_i$ while $\mathbf{e}_i \not\in C_j$, for $j \neq i$. Then for any unit $u \in \mathbb{C}$:

$$\Phi_{20u}(\mathbf{e}_0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u^{-2} & 0 & 0 \\ 0 & 0 & u^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Phi_{21u}(\mathbf{e}_1) = \begin{bmatrix} 0 & -u^{-2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & u^2 & 0 \end{bmatrix},$$

Canonical Basis

- $$\Phi_{22u}(\mathbf{e}_2) = \begin{bmatrix} 0 & 0 & -u^2 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & u^{-2} & 0 & 0 \end{bmatrix},$$
  $$\Phi_{23u}(\mathbf{e}_3) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -u^2 & 0 \\ 0 & -u^{-2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

All matrices above are unitary

Bell Basis

- Example
  The i-th vector $\mathbf{b}_i$ in the Bell basis of $\mathbb{H}_2$ is a maximally entangled 2-quregister:

$$\mathbf{b}_{2i_1+i_0} = \frac{1}{\sqrt{2}}\left(\mathbf{e}_0 \otimes \mathbf{e}_{i_1} + (-1)^{i_0}\mathbf{e}_1 \otimes \mathbf{e}_{1+i_1}\right).$$

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$$\Phi_{21u}(\mathbf{b}_3) = \Phi_{22u}(\mathbf{b}_3) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -u^{-2} & u^2 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -u^{-2} & u^2 & 0 \end{bmatrix}.$$

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- The spectral norm of B is $\|B\|_2 = \sqrt{2}$.

Measure of entanglement

- Proposition:
  For any $k \in \{0, 1, 2, 3\}$, any unitary complex number $u \in \mathbb{C}$ and any $\mathbf{x} \in C_k$:

$$\mathbf{x} \in Sp_2 \iff \Phi_{2ku}(\mathbf{x}) \in SU(2^2) \iff \nu_k(\mathbf{x}) = 0.$$

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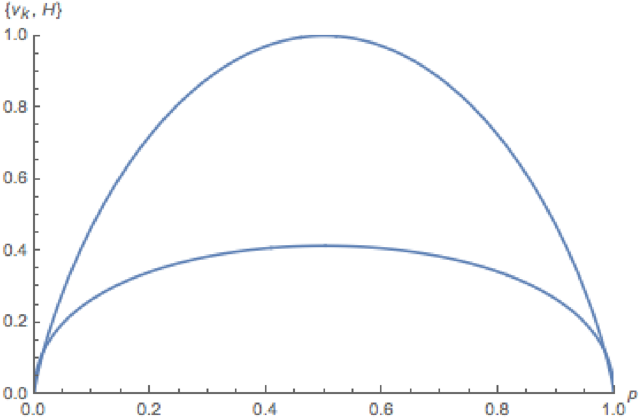
- The map ν_k can be considered a measure of entanglement,
  - Separability: If $\mathbf{x} \in S_{2^2}$ is separable, then $\nu_k(\mathbf{x}) = 0$.
  - Normality: In the maximally entangled vectors, ν_k attains its maxima. Indeed, the measure $\frac{2}{\sqrt{2}-1}\nu_k$ has $2 = \log_2 2^2$ as maximal value.

Measure of entanglement

- And
  - Continuity: ν_k is continuous with respect to the topology of S_{2^2}.
  - Boundedness under local operations: Any $\mathbf{x} \in S_{2^2}$ and any unitary operator $U : \mathbb{H}_2 \to \mathbb{H}_2$, $\nu_k(U\mathbf{x}) \leq \nu_k(\mathbf{x})$.

von Neumann entropy-ν



Graphs of H and ν_0. The maxima are $H(\frac{1}{2}) = 1$ and $\nu_0(\frac{1}{2}) = \sqrt{2} - 1$.

Conclusions

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- However, the proposed operators satisfy the desired embedding when they are restricted to the separable *n*-quregisters. These operators give rise to entanglement measures which are compatible with conventional entanglement measures, as von Neumann entropy.
- The procedures used in this work are rather standard and most probably can be generalised to the quregisters of any length. We look towards to formally prove this sketch of research.

Thank you!

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